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Asymptotic profiles for Choquard equations with combined attractive nonlinearities

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Abstract

We study asymptotic behaviour of positive ground state solutions of the nonlinear Choquard equation

$$-\Delta u + \varepsilon u = (I_\alpha * |u|^p)|u|^{p-2}u + |u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \tag{P_ε}$$

where $N \geq 3$ is an integer, $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$, $q \in (2, \frac{2N}{N-2})$, I_α is the Riesz potential of order $\alpha \in (0, N)$ and $\varepsilon > 0$ is a parameter. We show that as $\varepsilon \to 0$ (resp. $\varepsilon \to \infty$), the ground state solutions of (P_ε), after appropriate rescalings dependent on parameter regimes, converge in $H^1(\mathbb{R}^N)$ to particular solutions of five different limit equations. We also establish a sharp asymptotic characterisation of such rescalings, and the precise asymptotic behaviour of $u_\varepsilon(0)$, $\|\nabla u_\varepsilon\|_2^2$, $\|u_\varepsilon\|_2^2$, $\int_{\mathbb{R}^N}(I_\alpha * |u_\varepsilon|^p)|u_\varepsilon|^p$ and $\|u_\varepsilon\|_q^q$, which depend in a non-trivial way on the exponents p, q and the space dimension N. Further, we discuss a connection of our results with a mass constrained problem, associated to (P_ε) with normalization constraint $\int_{\mathbb{R}^N}|u|^2 = c^2$. As a consequence of the main results, we obtain the existence, multiplicity and precise asymptotic behaviour of positive normalized solutions of such a problem as $c \to 0$ and $c \to \infty$.

MSC: primary 35J60, 35Q55; secondary 35B25, 35B40, 35R09, 35J91

Keywords: Nonlinear Choquard equation; Ground state solution; Normalized solution; Concentration compactness; Asymptotic behaviour

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1. Introduction

We study standing wave solutions of the nonlinear Schrödinger equation with attractive combined nonlinearity

$$i\psi_t = \Delta\psi + (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi + a|\psi|^{q-2}\psi, \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where $N \geq 3$ is an integer, $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$, $q \in (2, 2^*)$ with $2^* = \frac{2N}{N-2}$, and I_α is the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha(N)}{|x|^{N-\alpha}}, \quad A_\alpha(N) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha},$$

where Γ denotes the Gamma function.

A theory of NLS with local combined power nonlinearities was developed by Tao, Visan and Zhang [45] and attracted a lot attention during the past decade (cf. [1,2,7,10,19,24–27,34,36,51] and further references therein). Nonlocal equation (1.1) in the case $p = 2$ and $\alpha = 2$ was proposed in cosmology, under the name of the Gross–Pitaevskii–Poisson equation, as a model to describe the dynamics of the Cold Dark Matter in the form of the Bose–Einstein Condensate [4,8,47]. In this model the nonlocal convolution term in (1.1) represents the Newtonian gravitational attraction between bosonic particles. The local term takes into account the short–range self–interaction between bosons. The non-interacting case $a = 0$ corresponds to the Schrödinger–Newton (Choquard) model of self–gravitating bosons [40], which is mathematically well–studied [38]. When $a = 1$ the quantum self–interaction between bosons is focusing/attractive, while for $a = -1$ the self–interaction is defocusing/repulsive, see surveys [9,39] for the astrophysical background. Mathematically, the repulsive case $a = -1$ was recently studied in [34], see also further references therein. In this work we are concerned with the attractive case $a = 1$.

A standing wave solutions of (1.1) with a frequency $\varepsilon > 0$ is a finite energy solution in the form

$$\psi(t, x) = e^{-i\varepsilon t}u(x).$$

In the case $a = 1$, this ansatz yields the equation for u in the form

$$-\Delta u + \varepsilon u = (I_\alpha * |u|^p)|u|^{p-2}u + |u|^{q-2}u, \quad \text{in } \mathbb{R}^N. \tag{P_ε}$$

A solution of (P_ε) is a critical point of the Action functional defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} |u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q. \tag{1.2}$$

A *ground state* of (P_ε) is a nontrivial critical point of I with a minimal energy amongst all nontrivial critical points of I.

The existence and qualitative properties of ground states $u_\varepsilon \in H^1(\mathbb{R}^N)$ to (P_ε) for every $\varepsilon > 0$ have been studied in [24,25] (see also Theorems A, B, C below). In this work we are interested in the limit asymptotic profile of the ground states u_ε of the problem (P_ε), and in the asymptotic behaviour of different norms of u_ε, as $\varepsilon \to 0$ and $\varepsilon \to \infty$. Of particular importance is the L^2*–mass* of the ground state

$$M(\varepsilon) := \|u_\varepsilon\|_2^2, \tag{1.3}$$

which plays a key role in the analysis of stability of the corresponding standing wave solution of the time–dependent NLS (1.1). The importance of $M(\varepsilon)$ is for instance seen in the Grillakis-Shatah-Strauss theory [15,16,41,50] of stability for these solutions within the time-dependent Schrödinger equation. The latter says that the solution u_ε is orbitally stable when $M'(\varepsilon) > 0$ and that it is unstable when $M'(\varepsilon) < 0$. Therefore the intervals where $M(\varepsilon)$ is increasing furnish stable solutions whereas those where $M(\varepsilon)$ is decreasing correspond to unstable solutions. The Grillakis-Shatah-Strauss theory relies on another conserved quantity, the energy, which is defined below and for which the variations of $M(\varepsilon)$ also play a crucial role.

Alternatively to the ground states with a prescribed frequency, one can search for standing wave solutions of (1.1) with a prescribed mass, and in this case the frequency is part of the unknown. That is, for a fixed $c > 0$, search for $u \in H^1(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ satisfying

$$\begin{cases} -\Delta u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u + |u|^{q-2}u, & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^2 = c^2. \end{cases} \tag{1.4}$$

A solution of (1.4) is a pair (u, λ) called a *normalized solution*. Here $\lambda \in \mathbb{R}$ arises as an a-priori unknown Lagrange multiplier. This approach seems to be particularly meaningful from the physical point of view, and often offers a good insight into the dynamical properties of the standing wave solutions for (1.1), such as stability or instability. It is standard to see that critical points of the Energy functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q \tag{1.5}$$

restricted to the (mass) constraint

$$S(c) = \{u \in H^1(\mathbb{R}^N) : \ \|u\|_2^2 = c^2\} \tag{1.6}$$

are normalised solution of (1.4), and every normalised solution of (1.4) is a solution of (P_ε) with $\varepsilon = -\lambda$. We refer the readers to [26,27,51] and the references therein.

In [36], the second author and C. Muratov studied the asymptotic properties of ground states for a class of scalar field equations with a defocusing exponent p and a focusing larger exponent q. More precisely, the following equation

$$-\Delta u + \varepsilon u = |u|^{p-2}u - |u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

is discussed in [36], where $N \geq 3$, $q > p > 2$. Later, in [23], M. Lewin and S. Rota Nodari prove a general result about the uniqueness and non-degeneracy of positive radial solutions to the above equation. The non-degeneracy of the unique solution u_ε allows the authors to derive its behaviour in the two limits $\varepsilon \to 0$ and $\varepsilon \to \varepsilon_*$, where ε_* is a threshold for the existence. Amongst other things, a precise asymptotic expression of $M(\varepsilon) = \|u_\varepsilon\|_2^2$ is obtained in [23]. This implies the uniqueness of energy minimizers at fixed mass in certain regimes.

In [34], Zeng Liu and the second author extend the results in [36] to a class of Choquard type equation

$$-\Delta u + \varepsilon u = (I_\alpha * |u|^p)|u|^{p-2}u - |u|^{q-2}u, \quad \text{in } \mathbb{R}^N. \tag{1.8}$$

Under near optimal assumptions on the exponents p and q, the limit profiles of the ground states are discussed in the two cases $\varepsilon \to 0$ and $\varepsilon \to \infty$. But the precise asymptotic behaviour of the mass of the ground states was not studied in [34].

The nonlinear Schrödinger equation with two focusing exponents p and q,

$$-\Delta u + \varepsilon u = |u|^{p-2}u + \mu|u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.9}$$

where $N \geq 3$, $2 < q < p \leq 2^*$ and $\mu > 0$ is a parameter, was considered in [1,2] by T. Akahori et al. When $\mu = 1$, $p = 2^*$ and $q \in (2, 2^*)$, the authors in [2] proved that for small $\varepsilon > 0$ the ground state is unique and as $\varepsilon \to 0$, the unique ground state u_ε tends to the unique positive solution of the equation $-\Delta u + u = u^{q-1}$. After a suitable rescaling, authors in [1] establish a uniform decay estimate for the ground state u_ε, and then prove the uniqueness and nondegeneracy of ground states u_ε for $N \geq 5$ and large $\varepsilon > 0$, and show that for $N \geq 3$, as $\varepsilon \to \infty$, u_ε tends to a particular solution of the critical Emden–Fowler equation. More recently, Jeanjean, Zhang and Zhong [22] also studied the asymptotic behaviour of solutions as $\varepsilon \to 0$ and $\varepsilon \to \infty$ for the equation (1.9) with a general subcritical nonlinearity and discussed the connection to the existence, non-existence and multiplicity of prescribed mass positive solutions to (1.9) with the associated L^2 constraint condition $\int_{\mathbb{R}^N} |u|^2 = c^2$. For other related papers, we refer the reader to [35] and the references therein.

For quite a long time paper [18] was the only one dealing with existence of normalized solutions in cases when the energy is unbounded from below on the L^2-constraint. More recently, however, problems of this type received much attention. We refer the readers to [43,44,48,49] and references therein for the existence and multiplicity of normalized solutions to the equations (1.9). In [48], Wei and Wu studied the existence and asymptotic behaviour of normalized solutions for (1.9) with $p = 2^*$, and obtained a precise asymptotic behaviour of ground states and mountain pass solutions as $\mu \to 0$ and μ goes to its upper bound, where $\lambda := -\varepsilon$ arises as a Lagrange multiplier. We refer the readers to [20,22,43,44,49] for the asymptotic behaviour of normalized solutions when the parameter μ varies in its range. Roughly speaking, the parameter μ modifies thresholds for the existence but does not change the qualitative properties of solutions. In a sense, changing the parameter μ is equivalent to changing the mass $c > 0$. More precisely, it follows from the reduction given in [49] that finding a normalized solution of (1.9) with $p = 2^*$ and mass constrained condition $\int_{\mathbb{R}^N} |u|^2 = c^2$ is equivalent to finding a normalized solution of the problem

$$\begin{cases} -\Delta u + \varepsilon u = |u|^{2^*-2}u + |u|^{q-2}u, \ \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^2 = c^2 \mu^{\frac{2}{q - q\gamma_q}}, \end{cases} \tag{1.10}$$

where $\gamma_q = \frac{N(q-2)}{2q}$. In particular, sending $\mu \to 0$ (resp. $\mu \to \infty$) is equivalent to sending the mass $c \to 0$ (resp. $c \to \infty$).

In [20], taking the mass $c > 0$ as a parameter, Jeanjean and Le also discuss the asymptotic behaviour of normalized solutions of (1.9) with $p = 2^*$. In the case $N \geq 4$, $2 < q < 2 + \frac{4}{N}$, amongst other things, Jeanjean and Le obtained a normalized solution u_c of mountain pass type for small $c > 0$ and proved that

$$\lim_{c\to 0} \|\nabla u_c\|_2^2 = S^{\frac{N}{2}}, \quad \lim_{c\to 0} E(u_c) = \frac{1}{N} S^{\frac{N}{2}}. \tag{1.11}$$

The relationship between action ground state and energy ground state is discussed in [11,17,20,21,49]. In particular, in these works it is shown that the energy ground state is necessarily an action ground state. So some of the results in the present paper can be used for the analysis of the asymptotic behaviour of normalized ground states. However the connection between normalized solutions of mountain pass type and action ground states is less understood. We shall address this problem in a forthcoming paper.

Organization of the paper. In Section 2, we state the main results in this paper. In Section 3, we give some preliminary results which are needed in the proof of our main results. Sections 4 and 5 are devoted to the proofs of Theorems 2.1 and 2.2, respectively. Finally, in the last section, we prove Theorem 2.3 and 2.4, and present some further results.

Basic notations. Throughout this paper, we assume $N \geq 3$. B_r denotes the ball in $\mathbb{R}^N$ with radius $r > 0$ and centred at the origin, $|B_r|$ and B_r^c denote its Lebesgue measure and its complement in $\mathbb{R}^N$, respectively.

$C_c^\infty(\mathbb{R}^N)$ is the space of the functions infinitely differentiable with compact support in $\mathbb{R}^N$.

$L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$ is the Lebesgue space with the norm $\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p\right)^{1/p}$.

$H^1(\mathbb{R}^N)$ is the usual Sobolev space with norm $\|u\|_{H^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\right)^{1/2}$.

$H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric}\}$.

$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$.

As usual, C, c, etc., denote generic positive constants. For any small $\epsilon > 0$ and two nonnegative functions $f(\epsilon)$ and $g(\epsilon)$, we write:

(1) $f(\epsilon) \lesssim g(\epsilon)$ or $g(\epsilon) \gtrsim f(\epsilon)$ if there exists a positive constant C independent of ϵ such that $f(\epsilon) \leq Cg(\epsilon)$.

(2) $f(\epsilon) \sim g(\epsilon)$ if $f(\epsilon) \lesssim g(\epsilon)$ and $f(\epsilon) \gtrsim g(\epsilon)$.

If $|f(\epsilon)| \lesssim |g(\epsilon)|$, we write $f(\epsilon) = O((g(\epsilon))$. We also denote by $\Theta = \Theta(\epsilon)$ a generic positive function satisfying $C_1\epsilon \leq \Theta(\epsilon) \leq C_2\epsilon$ for some positive numbers $C_1, C_2 > 0$, which are independent of ϵ. Finally, if $\lim f(\epsilon)/g(\epsilon) = 1$ as $\epsilon \to \epsilon_0$, then we write $f(\epsilon) \simeq g(\epsilon)$ as $\epsilon \to \epsilon_0$.

2. Main results

Consider the family of rescalings

$$v(x) = \varepsilon^s u(\varepsilon^t x).$$

It is easy to see that if we choose $s = -\frac{2+\alpha}{4(p-1)}$ and $t = -\frac{1}{2}$ then (P_ε) transforms to the equation

$$-\Delta v + v = (I_\alpha * |v|^p)|v|^{p-2}v + \varepsilon^{\Lambda_1}|v|^{q-2}v,$$

where

$$\Lambda_1 = \frac{q(2+\alpha) - 2(2p+\alpha)}{4(p-1)} \begin{cases} > 0, & \text{if } q > \frac{2(2p+\alpha)}{2+\alpha}, \\ = 0, & \text{if } q = \frac{2(2p+\alpha)}{2+\alpha}, \\ < 0, & \text{if } q < \frac{2(2p+\alpha)}{2+\alpha}. \end{cases}$$

If we choose $s = -\frac{1}{q-2}$ and $t = -\frac{1}{2}$ then (P_ε) transforms to the equation

$$-\Delta v + v = \varepsilon^{\Lambda_2}(I_\alpha * |v|^p)|v|^{p-2}v + |v|^{q-2}v,$$

where

$$\Lambda_2 = \frac{2(2p+\alpha) - q(2+\alpha)}{2(q-2)} \begin{cases} < 0, & \text{if } q > \frac{2(2p+\alpha)}{2+\alpha}, \\ = 0, & \text{if } q = \frac{2(2p+\alpha)}{2+\alpha}, \\ > 0, & \text{if } q < \frac{2(2p+\alpha)}{2+\alpha}. \end{cases}$$

Motivated by this scaling consideration, in what follows we consider the equations

$$-\Delta v + v = (I_\alpha * |v|^p)|v|^{p-2}v + \lambda|v|^{q-2}v, \quad \text{in } \mathbb{R}^N, \tag{Q_λ}$$

and

$$-\Delta v + v = \mu(I_\alpha * |v|^p)|v|^{p-2}v + |v|^{q-2}v, \quad \text{in } \mathbb{R}^N, \tag{Q_μ}$$

where $\lambda, \mu > 0$ are parameters and we assume $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$, $q \in (2, 2^*]$. It is well-known that with these assumptions on the powers p and q the problems (Q_λ) and (Q_μ) are variationally well-posed in $H^1(\mathbb{R}^N)$, and the corresponding energy functionals, defined by

$$I_\lambda(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p - \frac{\lambda}{q}\int_{\mathbb{R}^N} |v|^q$$

and

$$I_\mu(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2 - \frac{\mu}{2p}\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p - \frac{1}{q}\int_{\mathbb{R}^N} |v|^q,$$

respectively, are of class $\mathcal{C}^1$ on $H^1(\mathbb{R}^N)$. The ground states energies given by

$$m_\lambda := \inf_{v \in \mathcal{M}_\lambda} I_\lambda(v) \quad \text{and} \quad m_\mu := \inf_{v \in \mathcal{M}_\mu} I_\mu(v)$$

are well-defined, here $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ denote the corresponding Nehari manifolds

$$\mathcal{M}_\lambda := \left\{ v \in H^1(\mathbb{R}^N) \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2 = \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p + \lambda \int_{\mathbb{R}^N} |v|^q \right\},$$

$$\mathcal{M}_\mu := \left\{ v \in H^1(\mathbb{R}^N) \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2 = \mu \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p + \int_{\mathbb{R}^N} |v|^q \right\}.$$

The ground state solutions of (Q_λ) and (Q_μ) will be denoted by v_λ and v_μ respectively. The existence of these kind of solutions is proved in [24,25]. In particular, the following theorems are proved in [25].

Theorem A. *Let $N \geq 3$, $\alpha \in (0, N)$, $p = \frac{N+\alpha}{N}$ and $\lambda > 0$. Then there is a constant $\lambda_0 > 0$ such that (Q_λ) admits a positive ground state $v_\lambda \in H^1(\mathbb{R}^N)$ which is radially symmetric and radially nonincreasing if one of the following conditions holds:*

(1) $q \in (2, 2 + \frac{4}{N})$;

(2) $q \in [2 + \frac{4}{N}, 2^*)$ and $\lambda > \lambda_0$.

Theorem B. *Let $N \geq 3$, $\alpha \in (0, N)$, $p = \frac{N+\alpha}{N-2}$ and $\lambda > 0$. Then there is a constant $\lambda_1 > 0$ such that (Q_λ) admits a positive ground state $v_\lambda \in H^1(\mathbb{R}^N)$ which is radially symmetric and radially nonincreasing if one of the following conditions holds:*

(1) $N \geq 4$ and $q \in (2, 2^*)$;

(2) $N = 3$ and $q \in (4, 2^*)$;

(3) $N = 3$, $q \in (2, 4]$ and $\lambda > \lambda_1$.

Theorem C. *Let $N \geq 3$, $\alpha \in (0, N)$, $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right)$, $q \in (2, 2^*)$ and $\lambda > 0$. Then (Q_λ) admits a positive ground state $v_\lambda \in H^1(\mathbb{R}^N)$ which is radially symmetric and radially nonincreasing.*

The exponents $p = \frac{N+\alpha}{N}$ and $p = \frac{N+\alpha}{N-2}$ are known in the literature as *lower* and *upper critical exponents* for the Choquard type equations, respectively. The case covered in Theorem C is known as Choquard subcritical.

We are interested in the asymptotic behaviour and limit profiles of the ground states v_λ when λ is small or large.

In lower critical case $p = \frac{N+\alpha}{N}$, and when $\lambda > 0$ is small, we are going to show that after a suitable rescaling the limit equation for (Q_λ) is given by the critical Hardy-Littlewood-Sobolev equation

$$U = (I_\alpha * |U|^{\frac{N+\alpha}{N}})U^{\frac{\alpha}{N}}, \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

It is well-known [31], that the radial ground states of (2.1) are given by the function

$$U_1(x) := \left(\frac{A}{1 + |x|^2}\right)^{\frac{N}{2}} \tag{2.2}$$

with a suitable constant $A > 0$, and the family of its rescalings

$$U_\rho(x) := \rho^{-\frac{N}{2}} U_1(x/\rho), \quad \rho > 0. \tag{2.3}$$

In the lower critical case we prove the following.

Theorem 2.1. *If $p = \frac{N+\alpha}{N}$, $q \in (2, 2 + \frac{4}{N})$, and $\{v_\lambda\}$ is a family of radial ground states of (Q_λ), then for small $\lambda > 0$*

$$v_\lambda(0) \sim \lambda^{\frac{N}{4-N(q-2)}},$$
$$\|\nabla v_\lambda\|_2^2 \sim \lambda^{\frac{4}{4-N(q-2)}}, \quad \|v_\lambda\|_q^q \sim \lambda^{\frac{N(q-2)}{4-N(q-2)}},$$
$$\int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^{\frac{N+\alpha}{N}})|v_\lambda|^{\frac{N+\alpha}{N}} = S_1^{\frac{N+\alpha}{\alpha}} + O(\lambda^{\frac{4}{4-N(q-2)}}),$$
$$\|v_\lambda\|_2^2 = S_1^{\frac{N+\alpha}{\alpha}} + O(\lambda^{\frac{4}{4-N(q-2)}}).$$

Moreover, there exists $\zeta_\lambda \in (0, +\infty)$ verifying

$$\zeta_\lambda \sim \lambda^{-\frac{2}{4-N(q-2)}}$$

such that for small $\lambda > 0$, the rescaled family of ground states

$$w_\lambda(x) = \zeta_\lambda^{\frac{N}{2}} v_\lambda(\zeta_\lambda x)$$

satisfies

$$\|\nabla w_\lambda\|_2^2 \sim \|w_\lambda\|_q^q \sim \int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^{\frac{N+\alpha}{N}})|w_\lambda|^{\frac{N+\alpha}{N}} \sim \|w_\lambda\|_2^2 \sim 1,$$

and as $\lambda \to 0$, w_λ converges in $H^1(\mathbb{R}^N)$ to the extremal function U_{ρ_0}, where

$$\rho_0 = \left(\frac{2q\int_{\mathbb{R}^N} |\nabla U_1|^2}{N(q-2)\int_{\mathbb{R}^N} |U_1|^q}\right)^{\frac{2}{4-N(q-2)}}. \tag{2.4}$$

Furthermore, the least energy m_λ of the ground sate satisfies

$$\frac{\alpha}{2(N+\alpha)} S_1^{\frac{N+\alpha}{\alpha}} - m_\lambda \sim \lambda^{\frac{4}{4-N(q-2)}},$$

as $\lambda \to 0$, where

$$S_1 = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |u|^2}{(\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}}. \tag{2.5}$$

In the upper critical case $p = \frac{N+\alpha}{N-2}$ and when $\lambda > 0$ is small, we are going to show that after a suitable rescaling the limit equation for (Q_λ) is given by the critical Choquard equation

$$-\Delta V = (I_\alpha * |V|^{\frac{N+\alpha}{N-2}})V^{\frac{2+\alpha}{N-2}} \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

It is known [12] that the radial ground states of (2.6) are given by the function

$$V_1(x) := [N(N-2)]^{\frac{N-2}{4}} \left(\frac{1}{1+|x|^2}\right)^{\frac{N-2}{2}} \tag{2.7}$$

and the family of its rescalings

$$V_\rho(x) := \rho^{-\frac{N-2}{2}} V_1(x/\rho), \quad \rho > 0. \tag{2.8}$$

In the upper critical case we prove the following.

Theorem 2.2. *Let $p = \frac{N+\alpha}{N-2}$ and $q \in (2, 2^*)$, and $\{v_\lambda\}$ be a family of radial ground states of (Q_λ). If $N \geq 5$, then for small $\lambda > 0$*

$$v_\lambda(0) \sim \lambda^{-\frac{1}{q-2}},$$
$$\|v_\lambda\|_q^q \sim \lambda^{\frac{2N-q(N-2)}{(N-2)(q-2)}}, \quad \|v_\lambda\|_2^2 \sim \lambda^{\frac{4}{(N-2)(q-2)}},$$
$$\|\nabla v_\lambda\|_2^2 = S_\alpha^{\frac{N+\alpha}{2+\alpha}} + O(\lambda^{\frac{4}{(N-2)(q-2)}}),$$
$$\int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^{\frac{N+\alpha}{N-2}})|v_\lambda|^{\frac{N+\alpha}{N-2}} = S_\alpha^{\frac{N+\alpha}{2+\alpha}} + O(\lambda^{\frac{4}{(N-2)(q-2)}}).$$

Moreover, there exists $\zeta_\lambda \in (0, +\infty)$ verifying

$$\zeta_\lambda \sim \lambda^{\frac{2}{(N-2)(q-2)}},$$

such that for small $\lambda > 0$, the rescaled family of ground states

$$w_\lambda(x) = \zeta_\lambda^{\frac{N-2}{2}} v_\lambda(\zeta_\lambda x)$$

satisfies

$$\|\nabla w_\lambda\|_2^2 \sim \|w_\lambda\|_q^q \sim \int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^{\frac{N+\alpha}{N-2}})|w_\lambda|^{\frac{N+\alpha}{N-2}} \sim \|w_\lambda\|_2^2 \sim 1,$$

and as $\lambda \to 0$, w_λ converges in $H^1(\mathbb{R}^N)$ to V_{ρ_0} with

$$\rho_0 = \left(\frac{2(2^*-q)\int_{\mathbb{R}^N} |V_1|^q}{q(2^*-2)\int_{\mathbb{R}^N} |V_1|^2}\right)^{\frac{2}{(N-2)(q-2)}}. \tag{2.9}$$

In the lower dimension cases, we assume that $q \in (2, 4)$ if $N = 4$, and $q \in (4, 6)$ if $N = 3$, then for small $\lambda > 0$

$$v_\lambda(0) \sim \begin{cases} \lambda^{-\frac{1}{q-2}}(\ln\frac{1}{\lambda})^{\frac{1}{q-2}}, & \text{if } N=4,\\ \lambda^{-\frac{1}{q-4}}, & \text{if } N=3,\end{cases}$$

$$\|v_\lambda\|_q^q \sim \begin{cases} \lambda^{\frac{4-q}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}, & \text{if } N=4,\\ \lambda^{\frac{6-q}{q-4}}, & \text{if } N=3,\end{cases}$$

$$\|v_\lambda\|_2^2 \sim \begin{cases} \lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}, & \text{if } N=4,\\ \lambda^{\frac{2}{q-4}}, & \text{if } N=3,\end{cases}$$

$$\|\nabla v_\lambda\|_2^2 = S_\alpha^{\frac{N+\alpha}{2+\alpha}} + \begin{cases} O(\lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}), & \text{if } N=4,\\ O(\lambda^{\frac{2}{q-4}}), & \text{if } N=3,\end{cases}$$

$$\int_{\mathbb{R}^N}(I_\alpha * |v_\lambda|^{\frac{N+\alpha}{N-2}})|v_\lambda|^{\frac{N+\alpha}{N-2}} = S_\alpha^{\frac{N+\alpha}{2+\alpha}} + \begin{cases} O(\lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}), & \text{if } N=4,\\ O(\lambda^{\frac{2}{q-4}}), & \text{if } N=3,\end{cases}$$

and there exists $\zeta_\lambda \in (0,+\infty)$ *verifying*

$$\zeta_\lambda \sim \begin{cases} \lambda^{\frac{1}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{1}{q-2}}, & \text{if } N=4,\\ \lambda^{\frac{2}{q-4}}, & \text{if } N=3,\end{cases}$$

such that for small $\lambda>0$*, the rescaled family of ground states*

$$w_\lambda(x) = \zeta_\lambda^{\frac{N-2}{2}} v_\lambda(\zeta_\lambda x)$$

satisfies

$$\|\nabla w_\lambda\|_2^2 \sim \|w_\lambda\|_q^q \sim \int_{\mathbb{R}^N}(I_\alpha * |w_\lambda|^{\frac{N+\alpha}{N-2}})|w_\lambda|^{\frac{N+\alpha}{N-2}} \sim 1,$$

$$\|w_\lambda\|_2^2 \sim \begin{cases} \ln\frac{1}{\lambda}, & \text{if } N=4,\\ \lambda^{-\frac{2}{q-4}}, & \text{if } N=3,\end{cases}$$

and as $\lambda\to 0$*,* w_λ *converges in* $D^{1,2}(\mathbb{R}^N)$ *and* $L^q(\mathbb{R}^N)$ *to* V_1*. Furthermore, the least energy* m_λ *of the ground state satisfies*

$$\frac{2+\alpha}{2(N+\alpha)}S_\alpha^{\frac{N+\alpha}{2+\alpha}} - m_\lambda \sim \begin{cases} \lambda^{\frac{4}{(N-2)(q-2)}}, & \text{if } N\ge 5,\\ \lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}, & \text{if } N=4,\\ \lambda^{\frac{2}{q-4}}, & \text{if } N=3,\end{cases}$$

as $\lambda\to 0$*, where*

$$S_\alpha := \inf_{v\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla v|^2}{\left(\int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{N+\alpha}{N-2}})|v|^{\frac{N+\alpha}{N-2}}\right)^{\frac{N-2}{N+\alpha}}}. \tag{2.10}$$

Remark 2.1. If $N \geq 5$ and $\alpha > N-4$, we can choose $\zeta_\lambda = \lambda^{\frac{2}{(N-2)(q-2)}}$ in Theorem 2.2.

In the subcritical regime the limit equations are given by "formal" direct limits with no rescalings involved, both when $\lambda \to 0$ or $\mu \to 0$.

Theorem 2.3. *Let $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and $q \in (2, 2^*]$. Let v_λ be the radial ground state of (Q_λ), then for any sequence $\lambda_n \to 0$, there exists a subsequence, still denoted by λ_n, such that v_{λ_n} converges in $H^1(\mathbb{R}^N)$ to a positive solution $v_0 \in H^1(\mathbb{R}^N)$ of the equation*

$$-\Delta v + v = (I_\alpha * |v|^p)v^{p-1}. \tag{2.11}$$

Moreover, as $\lambda \to 0$, there holds

$$\|v_\lambda\|_2^2 = \frac{N+\alpha-p(N-2)}{2p} S_p^{\frac{p}{p-1}} + O(\lambda), \quad \text{if } q < \frac{2(2p+\alpha)}{2+\alpha},$$
$$\|v_\lambda\|_2^2 = \frac{N+\alpha-p(N-2)}{2p} S_p^{\frac{p}{p-1}} - \Theta(\lambda), \quad \text{if } q \geq \frac{2(2p+\alpha)}{2+\alpha},$$
$$\|\nabla v_\lambda\|_2^2 = \frac{N(p-1)-\alpha}{2p} S_p^{\frac{p}{p-1}} + O(\lambda),$$

and the least energy m_λ of the ground state satisfies

$$\frac{p-1}{2p} S_p^{\frac{p}{p-1}} - m_\lambda \sim \lambda,$$

as $\lambda \to 0$, where

$$S_p = \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2}{\left(\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p\right)^{\frac{1}{p}}}. \tag{2.12}$$

Theorem 2.4. *If $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$ and $q \in (2, 2^*)$. Let v_μ be the radial ground state of (Q_μ), then as $\mu \to 0$, v_μ converges in $H^1(\mathbb{R}^N)$ to the unique positive solution $v_0 \in H^1(\mathbb{R}^N)$ of the equation*

$$-\Delta v + v = v^{q-1}. \tag{2.13}$$

Moreover, as $\mu \to 0$, there holds

$$\|v_\mu\|_2^2 = \frac{2N-q(N-2)}{2q} S_q^{\frac{q}{q-2}} + O(\mu), \quad \text{if } q > \frac{2(2p+\alpha)}{2+\alpha},$$
$$\|v_\mu\|_2^2 = \frac{2N-q(N-2)}{2q} S_q^{\frac{q}{q-2}} - \Theta(\mu), \quad \text{if } q \leq \frac{2(2p+\alpha)}{2+\alpha},$$
$$\|\nabla v_\mu\|_2^2 = \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} + O(\mu),$$

$$\|u_\mu\|_q^q \sim \int_{\mathbb{R}^N} (I_\alpha * |u_\mu|^p)|u_\mu|^p \sim 1,$$

and the least energy m_μ of the ground state satisfies

$$\frac{q-2}{2q} S_q^{\frac{q}{q-2}} - m_\mu \sim \mu,$$

as $\mu \to 0$, where

$$S_q = \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2}{(\int_{\mathbb{R}^N} |v|^q)^{\frac{2}{q}}}. \tag{2.14}$$

Remark 2.2. The key novel results in the present paper are:

(I) In the lower critical case in Theorem 2.1 for all $N \geq 3$ and the upper critical case in Theorem 2.2 for $N \geq 5$, we obtain optimal explicit rescaling in a sense that it is unique up to a multiplicative constant such that the rescaled family of ground states converges in $H^1(\mathbb{R}^N)$ to a particular solution of the limit equation.

(II) This paper is inspired by [36], but the technique in the present paper is very different from that used in [36]. In [36], the second author and C. Muratov studied the asymptotic properties of ground states for a combined powers Schrödinger equation with a focusing exponent $p > 2$ and a defocusing exponent $q > p$. By considering a Berestycki-Lions type constrained minimization problem, the authors in [36] obtain a precise estimate of least energy which implies the uniform boundedness of the rescaled family of ground states in $L^q(\mathbb{R}^N)$. Berestycki-Lions constraint technique is not applicable in the nonlocal problems which involve multiple scaling regimes associated with the gradient and nonlocal parts of the problem. Instead, in the present paper, we first use the Nehari manifold and Pohožaev identity to obtain the uniform boundedness of the rescaled family of ground states in $L^q(\mathbb{R}^N)$ and then establish a precise estimate of least energy.

Connection with problem (P_ε). Converting equations (Q_λ) and (Q_μ) back to the original equation (P_ε) using the explicit rescalings introduced in the beginning of this section, we can deduce from Theorems A, B, C and Theorems 2.1–2.4 asymptotic properties of ground states of (P_ε). Figs. 1 and 2 below depict the limit equations of (P_ε) as $\varepsilon \to \infty$ and $\varepsilon \to 0$ when (p, q) belongs to different regions in the (p, q) plane, and reveal the asymptotic behaviour of rescaled family of ground states to (P_ε) as $\varepsilon \to \infty$ and $\varepsilon \to 0$, respectively.

In what follows we denote

$$M(0) := \lim_{\varepsilon\to 0} M(\varepsilon), \qquad M(\infty) := \lim_{\varepsilon\to\infty} M(\varepsilon).$$

The following three propositions give a summary of our main results in this paper, stated in terms of equation (P_ε). They are direct consequences of Theorems 2.1–2.4, Theorems A, B, C [24,25], and Lemma A.1, which is formulated and proved in the Appendix.

Proposition 2.1. *If $p = \frac{N+\alpha}{N}$ and $q \in (2, 2+\frac{4}{N})$, then the problem (P_ε) admits a positive ground state $u_\varepsilon \in H^1(\mathbb{R}^N)$, which is radially symmetric and radially nonincreasing. Furthermore, the following statements hold true:*

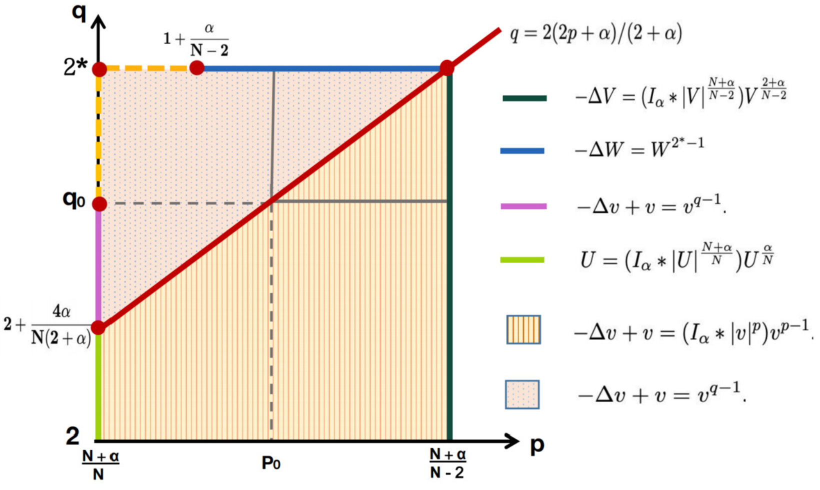

Fig. 1. The limit equations of (P_ε) as $\varepsilon\to\infty$. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

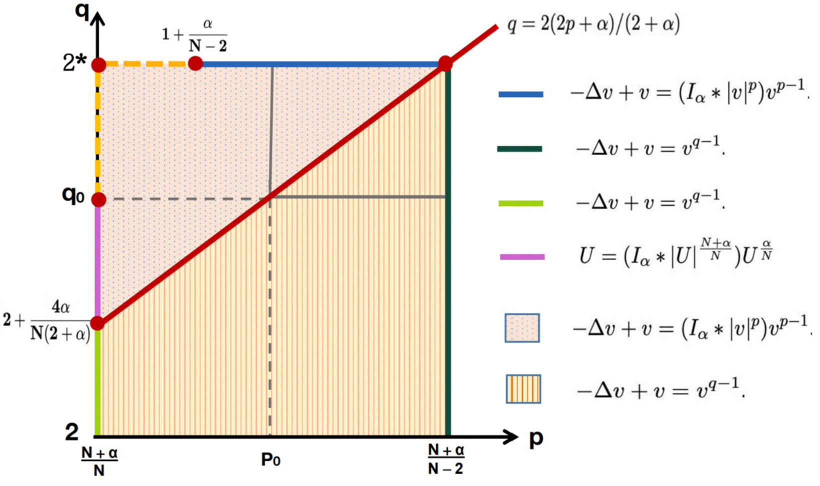

Fig. 2. The limit equations of (P_ε) as $\varepsilon\to 0$.

(I) As $\varepsilon\to 0$, there exists $\xi_\varepsilon\sim\varepsilon^{-\frac{N(q-2)}{\alpha[4-N(q-2)]}}$ such that the rescaled family of ground states

$$w_\varepsilon(x)=\begin{cases}\varepsilon^{-\frac{1}{q-2}}u_\varepsilon(\varepsilon^{-\frac{1}{2}}x), & \text{if}\quad q\in(2,2+\frac{4\alpha}{N(2+\alpha)}),\\ \varepsilon^{-\frac{N}{2\alpha}}\xi_\varepsilon^{\frac{N}{2}}u_\varepsilon(\xi_\varepsilon x), & \text{if}\quad q\in(2+\frac{4\alpha}{N(2+\alpha)},2+\frac{4}{N}),\end{cases}$$

converges in $H^1(\mathbb{R}^N)$ to the unique positive solution of the equation $-\Delta w+w=w^{q-1}$ if $q\in(2,2+\frac{4\alpha}{N(2+\alpha)})$ and converges in $H^1(\mathbb{R}^N)$ to the extremal function U_1 if $q\in(2+\frac{4\alpha}{N(2+\alpha)},2+\frac{4}{N})$. Moreover,

$$u_\varepsilon(0) \sim \begin{cases} \varepsilon^{\frac{1}{q-2}}, & \text{if} \quad q \in (2, 2+\frac{4\alpha}{N(2+\alpha)}], \\ \varepsilon^{\frac{2N}{\alpha[4-N(q-2)]}}, & \text{if} \quad q \in (2+\frac{4\alpha}{N(2+\alpha)}, 2+\frac{4}{N}), \end{cases}$$

$$\|u_\varepsilon\|_2^2 \sim \begin{cases} \varepsilon^{\frac{4-N(q-2)}{2(q-2)}} & \text{if} \quad q \in (2, 2+\frac{4\alpha}{N(2+\alpha)}], \\ \varepsilon^{\frac{N}{\alpha}} & \text{if} \quad q \in (2+\frac{4\alpha}{N(2+\alpha)}, 2+\frac{4}{N}), \end{cases}$$

$$\|\nabla u_\varepsilon\|_2^2 \sim \begin{cases} \varepsilon^{\frac{2N-q(N-2)}{2(q-2)}}, & \text{if} \quad q \in (2, 2+\frac{4\alpha}{N(2+\alpha)}], \\ \varepsilon^{\frac{N[2N-q(N-2)]}{\alpha[4-N(q-2)]}}, & \text{if} \quad q \in (2+\frac{4\alpha}{N(2+\alpha)}, 2+\frac{4}{N}), \end{cases}$$

$$E(u_\varepsilon) = \begin{cases} \varepsilon^{\frac{2N-q(N-2)}{2(q-2)}}\left[-\frac{4-N(q-2)}{4q}S_q^{\frac{q}{q-2}} + O(\varepsilon^{-\frac{N(2+\alpha)(q-2)-4\alpha}{2N(q-2)}})\right], & \text{if} \quad q \in (2, 2+\frac{4\alpha}{N(2+\alpha)}), \\ \varepsilon^{\frac{N+\alpha}{\alpha}}\left[-\frac{N}{2(N+\alpha)}S_1^{\frac{N+\alpha}{\alpha}} + O(\varepsilon^{\frac{N(2+\alpha)(q-2)-4\alpha}{\alpha[4-N(q-2)]}})\right], & \text{if} \quad q \in (2+\frac{4\alpha}{N(2+\alpha)}, 2+\frac{4}{N}). \end{cases}$$

(II) As $\varepsilon \to \infty$, there exists $\xi_\varepsilon \sim \varepsilon^{-\frac{N(q-2)}{\alpha[4-N(q-2)]}}$ such that the rescaled family of ground states

$$w_\varepsilon(x) = \begin{cases} \varepsilon^{-\frac{N}{2\alpha}}\xi_\varepsilon^{\frac{N}{2}}u_\varepsilon(\xi_\varepsilon x), & \text{if} \quad q \in (2, 2+\frac{4\alpha}{N(2+\alpha)}), \\ \varepsilon^{-\frac{1}{q-2}}u_\varepsilon(\varepsilon^{-\frac{1}{2}}x), & \text{if} \quad q \in (2+\frac{4\alpha}{N(2+\alpha)}, 2+\frac{4}{N}), \end{cases}$$

converges in $H^1(\mathbb{R}^N)$ to the extremal function U_1 if $q \in (2+\frac{4\alpha}{N(2+\alpha)}, 2+\frac{4}{N})$, and converges in $H^1(\mathbb{R}^N)$ to the unique positive solution of the equation $-\Delta w + w = w^{q-1}$ if $q \in (2, 2+\frac{4\alpha}{N(2+\alpha)})$. Moreover,

$$u_\varepsilon(0) \sim \begin{cases} \varepsilon^{\frac{2N}{\alpha[4-N(q-2)]}}, & \text{if} \quad q \in (2, 2+\frac{4\alpha}{N(2+\alpha)}], \\ \varepsilon^{\frac{1}{q-2}}, & \text{if} \quad q \in (2+\frac{4\alpha}{N(2+\alpha)}, 2+\frac{4}{N}), \end{cases}$$

$$\|u_\varepsilon\|_2^2 \sim \begin{cases} \varepsilon^{\frac{N}{\alpha}}, & \text{if} \quad q \in (2, 2+\frac{4\alpha}{N(2+\alpha)}], \\ \varepsilon^{\frac{4-N(q-2)}{2(q-2)}}, & \text{if} \quad q \in (2+\frac{4\alpha}{N(2+\alpha)}, 2+\frac{4}{N}), \end{cases}$$

$$\|\nabla u_\varepsilon\|_2^2 \sim \begin{cases} \varepsilon^{\frac{N[2N-q(N-2)]}{\alpha[4-N(q-2)]}}, & \text{if} \quad q \in (2, 2+\frac{4\alpha}{N(2+\alpha)}], \\ \varepsilon^{\frac{2N-q(N-2)}{2(q-2)}}, & \text{if} \quad q \in (2+\frac{4\alpha}{N(2+\alpha)}, 2+\frac{4}{N}), \end{cases}$$

$$E(u_\varepsilon) = \begin{cases} \varepsilon^{\frac{N+\alpha}{\alpha}}\left[-\frac{N}{2(N+\alpha)}S_1^{\frac{N+\alpha}{\alpha}} + O(\varepsilon^{\frac{N(2+\alpha)(q-2)-4\alpha}{\alpha[4-N(q-2)]}})\right], & \text{if} \quad q \in (2, 2+\frac{4\alpha}{N(2+\alpha)}), \\ \varepsilon^{\frac{2N-q(N-2)}{2(q-2)}}\left[-\frac{4-N(q-2)}{4q}S_q^{\frac{q}{q-2}} + O(\varepsilon^{-\frac{N(2+\alpha)(q-2)-4\alpha}{2N(q-2)}})\right], & \text{if} \quad q \in (2+\frac{4\alpha}{N(2+\alpha)}, 2+\frac{4}{N}). \end{cases}$$

(III) $M(0) = 0$, $M(+\infty) = +\infty$, and if $M(\varepsilon)$ is of class C^1 for small $\varepsilon > 0$ and large $\varepsilon > 0$, then there exist some small $\varepsilon_0 > 0$ and some large $\varepsilon_\infty > 0$ such that

$$M'(\varepsilon) > 0, \qquad \text{for all } \varepsilon \in (0, \varepsilon_0) \cup (\varepsilon_\infty, +\infty).$$

In the upper critical case we have the following.

Proposition 2.2. *If $p=\frac{N+\alpha}{N-2}$, $q\in(2,2^*)$ for $N\ge 4$ and $q\in(4,6)$ for $N=3$, then the problem (P_ε) admits a positive ground state $u_\varepsilon\in H^1(\mathbb{R}^N)$, which is radially symmetric and radially nonincreasing. Furthermore, the following statements hold true:*

(I) As $\varepsilon\to\infty$, there exists $\xi_\varepsilon\in(0,+\infty)$ verifying

$$\xi_\varepsilon\sim\begin{cases}\varepsilon^{-\frac{2}{(N-2)(q-2)}}, & \text{if}\quad N\ge 5,\ q\in(2,2^*),\\ (\varepsilon\ln\varepsilon)^{-\frac{1}{q-2}}, & \text{if}\quad N=4,\ q\in(2,4),\\ \varepsilon^{-\frac{1}{q-4}}, & \text{if}\quad N=3,\ q\in(4,6),\end{cases}$$

such that the rescaled family of ground states $w_\varepsilon(x)=\xi_\varepsilon^{\frac{N-2}{2}}u_\varepsilon(\xi_\varepsilon x)$ converges to V_1 in $H^1(\mathbb{R}^N)$ if $N\ge5$ and in $D^{1,2}(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ if $N=4,3$. Moreover,

$$u_\varepsilon(0)\sim\begin{cases}\varepsilon^{\frac{1}{q-2}}, & \text{if}\quad N\ge 5,\ q\in(2,2^*),\\ (\varepsilon\ln\varepsilon)^{\frac{1}{q-2}}, & \text{if}\quad N=4,\ q\in(2,4),\\ \varepsilon^{\frac{1}{2(q-4)}}, & \text{if}\quad N=3,\ q\in(4,6),\end{cases}$$

$$\|u_\varepsilon\|_2^2\sim\begin{cases}\varepsilon^{-\frac{4}{(N-2)(q-2)}}, & \text{if}\quad N\ge 5,\ q\in(2,2^*),\\ \varepsilon^{-\frac{2}{q-2}}(\ln\varepsilon)^{-\frac{4-q}{q-2}}, & \text{if}\quad N=4,\ q\in(2,4),\\ \varepsilon^{-\frac{q-2}{2(q-4)}}, & \text{if}\quad N=3,\ q\in(4,6),\end{cases}$$

$$\|\nabla u_\varepsilon\|_2^2=S_\alpha^{\frac{N+\alpha}{2+\alpha}}+\begin{cases}O(\varepsilon^{-\frac{2N-q(N-2)}{(N-2)(q-2)}}), & \text{if } N\ge 5,\ q\in(2,2^*),\\ O((\varepsilon\ln\varepsilon)^{-\frac{4-q}{q-2}}), & \text{if } N=4,\ q\in(2,4),\\ O(\varepsilon^{-\frac{6-q}{2(q-4)}}), & \text{if } N=3,\ q\in(4,6),\end{cases}$$

$$E(u_\varepsilon)=\frac{2+\alpha}{2(N+\alpha)}S_\alpha^{\frac{N+\alpha}{2+\alpha}}-\begin{cases}\Theta(\varepsilon^{-\frac{2N-q(N-2)}{(N-2)(q-2)}}), & \text{if } N\ge 5,\ q\in(2,2^*),\\ \Theta((\varepsilon\ln\varepsilon)^{-\frac{4-q}{q-2}}), & \text{if } N=4,\ q\in(2,4),\\ \Theta(\varepsilon^{-\frac{6-q}{2(q-4)}}), & \text{if } N=3,\ q\in(4,6).\end{cases}$$

(II) As $\varepsilon\to0$, the rescaled family of ground states $w_\varepsilon(x)=\varepsilon^{-\frac{1}{q-2}}u_\varepsilon(\varepsilon^{-\frac12}x)$ converges in $H^1(\mathbb{R}^N)$ to the unique positive solution of the equation $-\Delta w+w=w^{q-1}$. Moreover,

$$u_\varepsilon(0)\sim\varepsilon^{\frac{1}{q-2}},\qquad \text{if}\ \begin{cases}N\ge4,\ q\in(2,2^*),\\ N=3,\quad q\in(4,6),\end{cases}$$

$$\|u_\varepsilon\|_2^2\sim\varepsilon^{\frac{4-N(q-2)}{2(q-2)}},\quad \text{if}\ \begin{cases}N\ge4,\ q\in(2,2^*),\\ N=3,\quad q\in(4,6),\end{cases}$$

$$\|\nabla u_\varepsilon\|_2^2\sim\varepsilon^{\frac{2N-q(N-2)}{2(q-2)}},\quad \text{if}\ \begin{cases}N\ge4,\ q\in(2,2^*),\\ N=3,\quad q\in(4,6),\end{cases}$$

$$E(u_\varepsilon)=\varepsilon^{\frac{2N-q(N-2)}{2(q-2)}}\left[\frac{N(q-2)-4}{4q}S_q^{\frac{q}{q-2}}+O(\varepsilon^{\frac{(2+\alpha)(2N-q(N-2))}{2(q-2)}})\right].$$

(III) If $N \ge 4, q \in (2, 2+\frac{4}{N})$, then $M(0) = M(+\infty) = 0$, and if $M(\varepsilon)$ is of class C^1 for small $\varepsilon > 0$ and large $\varepsilon > 0$, then there exist some small $\varepsilon_0 > 0$ and large $\varepsilon_\infty > 0$ such that

$$M'(\varepsilon) > 0, \quad \text{for } \varepsilon \in (0, \varepsilon_0), \qquad M'(\varepsilon) < 0, \quad \text{for } \varepsilon \in (\varepsilon_\infty, +\infty).$$

If $N \ge 4, q \in (2+\frac{4}{N}, 2^*)$, or $N = 3, q \in (4, 6)$, then $M(0) = +\infty$, $M(+\infty) = 0$, and if $M(\varepsilon)$ is of class C^1 for small $\varepsilon > 0$ and large $\varepsilon > 0$, then there exist some small $\varepsilon_0 > 0$ and large $\varepsilon_\infty > 0$ such that

$$M'(\varepsilon) < 0, \qquad \text{for } \varepsilon \in (0, \varepsilon_0) \cup (\varepsilon_\infty, +\infty).$$

In the subcritical case we establish the following results.

Proposition 2.3. *If $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and $q \in (2, 2^*)$, then the problem (P_ε) admits a positive ground state $u_\varepsilon \in H^1(\mathbb{R}^N)$, which is radially symmetric and radially nonincreasing. Furthermore, let S_p and S_q be the constants given in Theorems 2.3 and 2.4, respectively, then the following statements hold true:*

(I) As $\varepsilon \to 0$, the rescaled family of ground states

$$w_\varepsilon(x) = \begin{cases} \varepsilon^{-\frac{1}{q-2}} u_\varepsilon(\varepsilon^{-\frac{1}{2}} x), & \text{if} \quad q < \frac{2(2p+\alpha)}{2+\alpha}, \\ \varepsilon^{-\frac{2+\alpha}{4(p-1)}} u_\varepsilon(\varepsilon^{-\frac{1}{2}} x), & \text{if} \quad q > \frac{2(2p+\alpha)}{2+\alpha}, \end{cases}$$

converges in $H^1(\mathbb{R}^N)$ (up to a subsequence) to a positive solution of the equation

$$\begin{cases} -\Delta w + w = w^{q-1}, & \text{if} \quad q < \frac{2(2p+\alpha)}{2+\alpha}, \\ -\Delta w + w = (I_\alpha * |w|^p) w^{p-1}, & \text{if} \quad q > \frac{2(2p+\alpha)}{2+\alpha}. \end{cases}$$

Moreover,

$$u_\varepsilon(0) \sim \begin{cases} \varepsilon^{\frac{1}{q-2}}, & \text{if} \quad q \le \frac{2(2p+\alpha)}{2+\alpha}, \\ \varepsilon^{\frac{2+\alpha}{4(p-1)}}, & \text{if} \quad q > \frac{2(2p+\alpha)}{2+\alpha}, \end{cases}$$

$$\|u_\varepsilon\|_2^2 \sim \begin{cases} \varepsilon^{\frac{4-N(q-2)}{2(q-2)}}, & \text{if} \quad q \le \frac{2(2p+\alpha)}{2+\alpha}, \\ \varepsilon^{\frac{2+\alpha-N(p-1)}{2(p-1)}}, & \text{if} \quad q > \frac{2(2p+\alpha)}{2+\alpha}, \end{cases}$$

and

$$\|\nabla u_\varepsilon\|_2^2 \simeq \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} \varepsilon^{\frac{2N-q(N-2)}{2(q-2)}}, \quad \text{if } q \ne \frac{2(2p+\alpha)}{2+\alpha}.$$

If $q \ne \frac{2(2p+\alpha)}{2+\alpha}$, then as $\varepsilon \to 0$

$$M(\varepsilon)=\begin{cases} \varepsilon^{\frac{4-N(q-2)}{2(q-2)}}\left(\frac{2N-q(N-2)}{2q}S_q^{\frac{q}{q-2}}-\Theta(\varepsilon^{\frac{2(2p+\alpha)-q(2+\alpha)}{2(q-2)}})\right), & if \quad q<\frac{2(2p+\alpha)}{2+\alpha},\\ \varepsilon^{\frac{2+\alpha-N(p-1)}{2(p-1)}}\left(\frac{N+\alpha-p(N-2)}{2p}S_p^{\frac{p}{p-1}}-\Theta(\varepsilon^{\frac{q(2+\alpha)-2(2p+\alpha)}{4(p-1)}})\right), & if \quad q>\frac{2(2p+\alpha)}{2+\alpha}, \end{cases}$$

$$E(u_\varepsilon)=\begin{cases} \varepsilon^{\frac{2N-q(N-2)}{2(q-2)}}\left(\frac{N(q-2)-4}{4q}S_q^{\frac{q}{q-2}}+O(\varepsilon^{\frac{2(2p+\alpha)-q(2+\alpha)}{2(q-2)}})\right), & if \quad q<\frac{2(2p+\alpha)}{2+\alpha},\\ \varepsilon^{\frac{N+\alpha-p(N-2)}{2(p-1)}}\left(\frac{N(p-1)-2-\alpha}{4p}S_p^{\frac{p}{p-1}}+O(\varepsilon^{\frac{q(2+\alpha)-2(2p+\alpha)}{4(p-1)}})\right), & if \quad q>\frac{2(2p+\alpha)}{2+\alpha}. \end{cases}$$

(II) As $\varepsilon\to\infty$, the rescaled family of ground states

$$w_\varepsilon(x)=\begin{cases} \varepsilon^{-\frac{2+\alpha}{4(p-1)}}u_\varepsilon(\varepsilon^{-\frac12}x), & if \quad q<\frac{2(2p+\alpha)}{2+\alpha},\\ \varepsilon^{-\frac{1}{q-2}}u_\varepsilon(\varepsilon^{-\frac12}x), & if \quad q>\frac{2(2p+\alpha)}{2+\alpha}, \end{cases}$$

converges in $H^1(\mathbb{R}^N)$ (up to a subsequence) to a positive solution of the equation

$$\begin{cases} -\Delta w+w=(I_\alpha*|w|^p)w^{p-1}, & if \quad q<\frac{2(2p+\alpha)}{2+\alpha},\\ -\Delta w+w=w^{q-1}, & if \quad q>\frac{2(2p+\alpha)}{2+\alpha}. \end{cases}$$

Moreover,

$$u_\varepsilon(0)\sim\begin{cases} \varepsilon^{\frac{2+\alpha}{4(p-1)}}, & if \quad q\le\frac{2(2p+\alpha)}{2+\alpha},\\ \varepsilon^{\frac{1}{q-2}}, & if \quad q>\frac{2(2p+\alpha)}{2+\alpha}, \end{cases}$$

$$\|u_\varepsilon\|_2^2\sim\begin{cases} \varepsilon^{\frac{2+\alpha-N(p-1)}{2(p-1)}}, & if \quad q\le\frac{2(2p+\alpha)}{2+\alpha},\\ \varepsilon^{\frac{4-N(q-2)}{2(q-2)}}, & if \quad q>\frac{2(2p+\alpha)}{2+\alpha}, \end{cases}$$

and

$$\|\nabla u_\varepsilon\|_2^2\simeq\frac{N(p-1)-\alpha}{2p}S_p^{\frac{p}{p-1}}\varepsilon^{\frac{N+\alpha-p(N-2)}{2(p-1)}}, \quad if \ q\neq\frac{2(2p+\alpha)}{2+\alpha}.$$

If $q\neq\frac{2(2p+\alpha)}{2+\alpha}$, then as $\varepsilon\to\infty$

$$M(\varepsilon)=\begin{cases} \varepsilon^{\frac{2+\alpha-N(p-1)}{2(p-1)}}\left(\frac{N+\alpha-p(N-2)}{2p}S_p^{\frac{p}{p-1}}+O(\varepsilon^{-\frac{2(2p+\alpha)-q(2+\alpha)}{4(p-1)}})\right), & if \quad q<\frac{2(2p+\alpha)}{2+\alpha},\\ \varepsilon^{\frac{4-N(q-2)}{2(q-2)}}\left(\frac{2N-q(N-2)}{2q}S_q^{\frac{q}{q-2}}+O(\varepsilon^{-\frac{q(2+\alpha)-2(2p+\alpha)}{2(q-2)}})\right), & if \quad q>\frac{2(2p+\alpha)}{2+\alpha}, \end{cases}$$

$$E(u_\varepsilon)=\begin{cases} \varepsilon^{\frac{N+\alpha-p(N-2)}{2(p-1)}}\left(\frac{N(p-1)-2-\alpha}{4p}S_p^{\frac{p}{p-1}}+O(\varepsilon^{-\frac{2(2p+\alpha)-q(2+\alpha)}{4(p-1)}})\right), & if \quad q<\frac{2(2p+\alpha)}{2+\alpha},\\ \varepsilon^{\frac{2N-q(N-2)}{2(q-2)}}\left(\frac{N(q-2)-4}{4q}S_q^{\frac{q}{q-2}}+O(\varepsilon^{-\frac{q(2+\alpha)-2(2p+\alpha)}{2(q-2)}})\right), & if \quad q>\frac{2(2p+\alpha)}{2+\alpha}. \end{cases}$$

(III) Let $p_0 := 1 + \frac{2+\alpha}{N}$ *and* $q_0 := 2 + \frac{4}{N}$*, then*

$$M(0) = \begin{cases} 0, & \text{if} \quad q < q_0 \ \text{or} \ p < p_0, \\ \frac{2}{N+2} S_{q_0}^{\frac{N+2}{2}}, & \text{if} \quad q = q_0 \ \text{and} \ p > p_0, \\ \frac{2+\alpha}{N+2+\alpha} S_{p_0}^{\frac{N+2+\alpha}{2+\alpha}}, & \text{if} \quad q > q_0 \ \text{and} \ p = p_0, \\ \infty, & \text{if} \quad q > q_0 \ \text{and} \ p > p_0, \end{cases}$$

and

$$M(\infty) = \begin{cases} 0, & \text{if} \quad q > q_0 \ \text{or} \ p > p_0, \\ \frac{2}{N+2} S_{q_0}^{\frac{N+2}{2}}, & \text{if} \quad q = q_0 \ \text{and} \ p < p_0, \\ \frac{2+\alpha}{N+2+\alpha} S_{p_0}^{\frac{N+2+\alpha}{2+\alpha}}, & \text{if} \quad q < q_0 \ \text{and} \ p = p_0, \\ \infty, & \text{if} \quad q < q_0 \ \text{and} \ p < p_0. \end{cases}$$

Moreover, if $q \neq \frac{2(2p+\alpha)}{2+\alpha}$ *and* $M(\varepsilon)$ *is of class* C^1 *for small* $\varepsilon > 0$ *and large* $\varepsilon > 0$*, then there exists a small* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$*,*

$$\begin{cases} M'(\varepsilon) > 0, & \text{if} \quad q < q_0 \ \text{or} \ p < p_0, \\ M'(\varepsilon) < 0, & \text{if} \quad q = q_0 \ \text{and} \ p > p_0 \\ M'(\varepsilon) < 0, & \text{if} \quad q > q_0 \ \text{and} \ p = p_0 \\ M'(\varepsilon) < 0, & \text{if} \quad q > q_0 \ \text{and} \ p > p_0, \end{cases} \tag{2.15}$$

and there exists a large $\varepsilon_\infty > 0$ *such that for any* $\varepsilon \in (\varepsilon_\infty, +\infty)$*,*

$$\begin{cases} M'(\varepsilon) < 0, & \text{if} \quad q > q_0 \ \text{or} \ p > p_0, \\ M'(\varepsilon) > 0, & \text{if} \quad q < q_0 \ \text{and} \ p < p_0. \end{cases} \tag{2.16}$$

Remark 2.3. From the asymptotic expressions for $M(\varepsilon)$ in Propositions 2.1–2.3, and under an additional assumption that $M(\varepsilon)$ is of class C^1 for small and large ε, the sign of $M'(\varepsilon)$ follows from Lemma A.1 in the Appendix. Note that by Proposition 2.3(I), as $\varepsilon \to 0$ we have

$$M(\varepsilon) = \frac{2}{N+2} S_{q_0}^{\frac{N+2}{2}} - \Theta(\varepsilon^{\frac{N(p-1)-2-\alpha}{2}}), \quad \text{if } q = q_0 \text{ and } p > p_0,$$

$$M(\varepsilon) = \frac{2+\alpha}{N+2+\alpha} S_{p_0}^{\frac{N+2+\alpha}{2+\alpha}} - \Theta(\varepsilon^{\frac{N(q-2)-4}{4}}), \quad \text{if } q > q_0 \text{ and } p = p_0.$$

Therefore, to prove the second and third inequalities in (2.15), we replace $M(\varepsilon)$ by

$$\frac{2}{N+2} S_{q_0}^{\frac{N+2}{2}} - M(\varepsilon) \quad \text{and} \quad \frac{2+\alpha}{N+2+\alpha} S_{p_0}^{\frac{N+2+\alpha}{2+\alpha}} - M(\varepsilon),$$

in Lemma A.1, respectively.

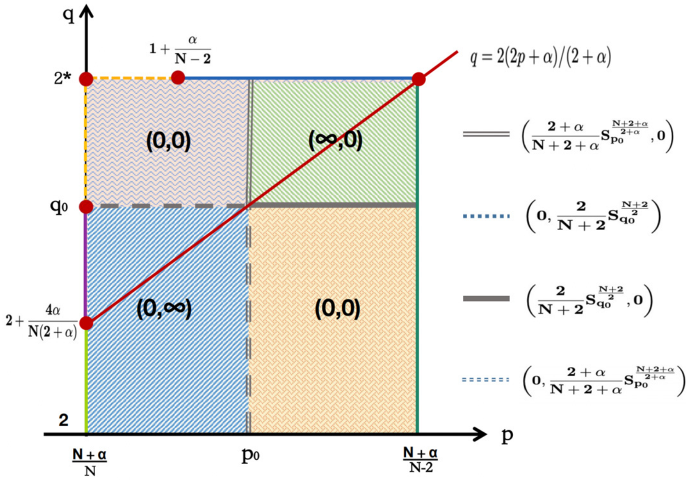

Fig. 3. The variation of $M(\varepsilon)$ for small and large ε, here $(\cdot,\cdot)=(M(0),M(\infty))$.

Fig. 3 outlines the limits of $M(\varepsilon)$ as $\varepsilon\to0$ and $\varepsilon\to\infty$ and reveals the variation of $M(\lambda)$ for small $\varepsilon>0$ and large $\varepsilon>0$ when (p,q) belongs to different regions in the (p,q) plane, as described in Propositions 2.1-2.3.

Connection with normalised solutions of (P_ε). It is clear that if $u_\varepsilon\in H^1(\mathbb{R}^N)$ is a ground state of (P_ε), and for some $c>0$ it holds

$$M(\varepsilon)=\|u_\varepsilon\|_2^2=c^2, \tag{2.17}$$

then u_ε is a normalized solution of (1.4) with $\lambda=-\varepsilon$. We denote this normalized solution by a pair (u_c,λ_c) with $\lambda_c=-\varepsilon$, or just u_c for simplicity. As direct consequences of Propositions 2.1–2.3, we deduce the following results.

Corollary 2.1. *Let $p=\frac{N+\alpha}{N}$, $q\in(2,2+\frac{4}{N})$, then for any $c>0$ the problem (1.4) has at least one positive normalized solution $u_c\in H^1(\mathbb{R}^N)$, which is radially symmetric and radially nonincreasing. Moreover, as $c\to0$,*

$$\|\nabla u_c\|_2^2\sim c^{\frac{2(2N-q(N-2))}{4-N(q-2)}}\to0,$$

$$E(u_c)\sim\begin{cases}-c^{\frac{2(2N-q(N-2))}{4-N(q-2)}}\to0^-, & \text{if}\quad q\in(2,2+\frac{4\alpha}{N(2+\alpha)}),\\ -c^{\frac{2(N+\alpha)}{N}}\to0^-, & \text{if}\quad q\in(2+\frac{4\alpha}{N(2+\alpha)},2+\frac{4}{N}).\end{cases}$$

As $c\to\infty$,

$$\|\nabla u_c\|_2^2\sim c^{\frac{2(2N-q(N-2))}{4-N(q-2)}}\to+\infty,$$

$$E(u_c)\sim\begin{cases}-c^{\frac{2(N+\alpha)}{N}}\to-\infty, & \text{if}\quad q\in(2,2+\frac{4\alpha}{N(2+\alpha)}),\\ -c^{\frac{2(2N-q(N-2))}{4-N(q-2)}}\to-\infty, & \text{if}\quad q\in(2+\frac{4\alpha}{N(2+\alpha)},2+\frac{4}{N}).\end{cases}$$

Corollary 2.2. *If $p = \frac{N+\alpha}{N-2}$, $q \in (2, 2^*)$ for $N \geq 4$ and $q \in (4, 6)$ for $N = 3$, then the following statements hold true:*

(I) If $q < q_0$, then there exists a constant $c_0 > 0$ such that for any $c \in (0, c_0)$, the problem (1.4) has at least two positive normalized solutions $u_c^1, u_c^2 \in H^1(\mathbb{R}^N)$, which are radially symmetric and radially nonincreasing. Moreover,

$$\|\nabla u_c^1\|_2^2 \sim c^{\frac{2(2N-q(N-2))}{4-N(q-2)}} \to 0, \quad \text{as } c \to 0,$$

$$E(u_c^1) \simeq \frac{4q}{N(q-2)} S_q^{\frac{q}{q-2}} c^{\frac{2(2N-q(N-2))}{4-N(q-2)}} \to 0^-, \quad \text{as } c \to 0,$$

$$\|\nabla u_c^2\|_2^2 \simeq S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \qquad E(u_c^2) \simeq \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \quad \text{as } c \to 0.$$

(II) If $q > q_0$, then for any $c > 0$ the problem (1.4) has at least one positive normalized solution $u_c \in H^1(\mathbb{R}^N)$, which is radially symmetric and radially nonincreasing. Moreover,

$$\|\nabla u_c\|_2^2 \simeq S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \qquad E(u_c) \simeq \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \quad \text{as } c \to 0,$$

$$\|\nabla u_c\|_2^2 \sim c^{\frac{2(2N-q(N-2))}{4-N(q-2)}} \to 0, \quad \text{as } c \to \infty,$$

$$E(u_c) \simeq \frac{4q}{N(q-2)} S_q^{\frac{q}{q-2}} c^{\frac{2(2N-q(N-2))}{4-N(q-2)}} \to 0^+, \quad \text{as } c \to \infty.$$

Corollary 2.3. *Let $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$, $q \in (2, 2^*)$, then the following statements hold true:*

(I) If $p < p_0, q < q_0$ or $p > p_0, q > q_0$, then for any $c > 0$ the problem (1.4) has at least one positive normalized solution $u_c \in H^1(\mathbb{R}^N)$. Moreover, if $p < p_0, q < q_0$ and $q < \frac{2(2p+\alpha)}{2+\alpha}$, then

$$\|\nabla u_c\|_2^2 \simeq \begin{cases} \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2 \right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to 0, & \text{as} \quad c \to 0, \\ \frac{N(p-1)-\alpha}{2p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2 \right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to +\infty, & \text{as} \quad c \to \infty, \end{cases}$$

$$E(u_c) \simeq \begin{cases} \frac{N(q-2)-4}{4q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2 \right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to 0^-, & \text{as} \quad c \to 0, \\ \frac{N(p-1)-2-\alpha}{4p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2 \right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to -\infty, & \text{as} \quad c \to \infty. \end{cases}$$

If $p < p_0, q < q_0$ and $q > \frac{2(2p+\alpha)}{2+\alpha}$, then

$$\|\nabla u_c\|_2^2 \simeq \begin{cases} \frac{N(p-1)-\alpha}{2p} S_p^{\frac{p}{p-1}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2 \right)^{\frac{(q-2)(N+\alpha-p(N-2))}{(p-1)(4-N(q-2))}} \to +\infty, & \text{as} \quad c \to \infty, \\ \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2 \right)^{\frac{(p-1)(2N-q(N-2))}{(q-2)(2+\alpha-N(p-1))}} \to 0, & \text{as} \quad c \to 0, \end{cases}$$

$$E(u_c) \simeq \begin{cases} \frac{N(q-2)-4}{4q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to -\infty, & as \quad c \to \infty, \\ \frac{N(p-1)-2-\alpha}{4p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to 0^-, & as \quad c \to 0. \end{cases}$$

If $p > p_0, q > q_0$ and $q < \frac{2(2p+\alpha)}{2+\alpha}$, then

$$\|\nabla u_c\|_2^2 \simeq \begin{cases} \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to 0, & as \quad c \to \infty, \\ \frac{N(p-1)-\alpha}{2p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to +\infty, & as \quad c \to 0, \end{cases}$$

$$E(u_c) \simeq \begin{cases} \frac{N(q-2)-4}{4q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to 0^+, & as \quad c \to \infty, \\ \frac{N(p-1)-2-\alpha}{4p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to +\infty, & as \quad c \to 0. \end{cases}$$

If $p > p_0, q > q_0$ and $q > \frac{2(2p+\alpha)}{2+\alpha}$, then

$$\|\nabla u_c\|_2^2 \simeq \begin{cases} \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{(p-1)(2N-q(N-2))}{(q-2)(2+\alpha-N(p-1))}} \to 0, & as \quad c \to \infty, \\ \frac{N(p-1)-\alpha}{2p} S_p^{\frac{p}{p-1}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{(q-2)(N+\alpha-p(N-2))}{(p-1)(4-N(q-2))}} \to +\infty, & as \quad c \to 0, \end{cases}$$

$$E(u_c) \simeq \begin{cases} \frac{N(p-1)-2-\alpha}{4p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to 0^+, & as \quad c \to \infty, \\ \frac{N(q-2)-4}{4q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to +\infty, & as \quad c \to 0. \end{cases}$$

(II) If $p = p_0$ and $q \neq q_0$ (resp. $q = q_0$ and $p \neq p_0$), then for any $c \in (0, \sqrt{\frac{2+\alpha}{N+2+\alpha}} S_{p_0}^{\frac{N+2+\alpha}{2(2+\alpha)}})$ (resp. $c \in (0, \sqrt{\frac{2}{N+2}} S_{q_0}^{\frac{N+2}{4}})$), the problem (1.4) has at least one positive normalized solution $u_c \in H^1(\mathbb{R}^N)$. Moreover, if $p = p_0, q < q_0$, then as $c \to 0$,

$$\|\nabla u_c\|_2^2 \simeq \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to 0,$$

$$E(u_c) \simeq \frac{N(q-2)-4}{4q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to 0^-.$$

If $p = p_0, q > q_0$, then as $c \to 0$,

$$\|\nabla u_c\|_2^2 \simeq \frac{N}{N+2+\alpha} S_{p_0}^{\frac{N+2+\alpha}{2+\alpha}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2(q-2)}{4-N(q-2)}} \to +\infty,$$

$$E(u_c) \simeq \frac{N(q-2)-4}{4q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to +\infty,$$

and as $c \to \sqrt{\frac{2+\alpha}{N+2+\alpha} S_{p_0}^{\frac{N+2+\alpha}{2(2+\alpha)}}}$,

$$\|\nabla u_c\|_2^2 \sim \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} \left(\frac{2+\alpha}{N+2+\alpha} S_{p_0}^{\frac{N+2+\alpha}{2+\alpha}} - c^2\right)^{\frac{2(2N-q(N-2))}{(q-2)(N(q-2)-4)}} \to 0,$$

$$E(u_c) = O\left(\left(\frac{2+\alpha}{N+2+\alpha} S_{p_0}^{\frac{N+2+\alpha}{2+\alpha}} - c^2\right)^{\frac{N(q-2)}{N(q-2)-4}} \right) \to 0.$$

If $p < p_0, q = q_0$*, then as* $c \to 0$*,*

$$\|\nabla u_c\|_2^2 \simeq \frac{N}{N+2} S_{q_0}^{\frac{N+2}{2}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{2(p-1)}{2+\alpha-N(p-1)}} \to 0,$$

$$E(u_c) \simeq \frac{N(p-1)-2-\alpha}{4p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to 0^-.$$

If $p > p_0, q = q_0$*, then as* $c \to 0$*,*

$$\|\nabla u_c\|_2^2 \simeq \frac{N(p-1)-\alpha}{2p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to +\infty,$$

$$E(u_c) \simeq \frac{N(p-1)-2-\alpha}{4p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to +\infty,$$

and as $c \to \sqrt{\frac{2}{N+2} S_{q_0}^{\frac{N+2}{4}}}$,

$$\|\nabla u_c\|_2^2 \sim \frac{N}{N+2} S_{q_0}^{\frac{N+2}{2}} \left(\frac{2}{N+2} S_{q_0}^{\frac{N+2}{2}} - c^2\right)^{\frac{2}{N(p-1)-2-\alpha}} \to 0,$$

$$E(u_c) = O\left(\left(\frac{2}{N+2} S_{q_0}^{\frac{N+2}{2}} - c^2\right)^{\frac{N(p-1)-\alpha}{N(p-1)-2-\alpha}} \right) \to 0.$$

(III) If $p < p_0, q > q_0$*, then for any* $c \in (0, \sup_{\varepsilon>0} \sqrt{M(\varepsilon)})$ *the problem (1.4) has two positive normalized solutions* u_c^1 *and* u_c^2 *satisfying*

$$\|\nabla u_c^1\|_2^2 \simeq \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{(p-1)(2N-q(N-2))}{(q-2)(2+\alpha-N(p-1))}} \to 0, \text{ as } c \to 0,$$

$$E(u_c^1) \simeq \frac{N(p-1)-2-\alpha}{4p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to 0^-, \ as\ c \to 0,$$

$$\|\nabla u_c^2\|_2^2 \simeq \frac{N(p-1)-\alpha}{2p} S_p^{\frac{p}{p-1}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{(q-2)(N+\alpha-p(N-2))}{(p-1)(4-N(q-2))}} \to +\infty, \ as\ c \to 0,$$

$$E(u_c^2) \simeq \frac{N(q-2)-4}{4q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to +\infty, \ as\ c \to 0.$$

If $p > p_0, q < q_0$, then for any $c \in (0, \sup_{\varepsilon>0} \sqrt{M(\varepsilon)})$, the problem (1.4) has two positive normalized solutions u_c^1 and u_c^2 satisfying

$$\|\nabla u_c^1\|_2^2 \simeq \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to 0, \ as\ c \to 0,$$

$$E(u_c^1) \simeq \frac{N(q-2)-4}{4q} S_q^{\frac{q}{q-2}} \left(\frac{2q}{2N-q(N-2)} S_q^{-\frac{q}{q-2}} c^2\right)^{\frac{2N-q(N-2)}{4-N(q-2)}} \to 0^-, \ as\ c \to 0,$$

$$\|\nabla u_c^2\|_2^2 \simeq \frac{N(p-1)-\alpha}{2p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to +\infty, \ as\ c \to 0,$$

$$E(u_c^2) \simeq \frac{N(p-1)-2-\alpha}{4p} S_p^{\frac{p}{p-1}} \left(\frac{2p}{N+\alpha-p(N-2)} S_p^{-\frac{p}{p-1}} c^2\right)^{\frac{N+\alpha-p(N-2)}{2+\alpha-N(p-1)}} \to +\infty, \ as\ c \to 0.$$

We note that some similar existence results on normalized solutions are already obtained in [26–29,51,52] in the whole possible range of parameters, but the precise asymptotic behaviour of the normalized solutions is not addressed there.

Remark 2.4. In the case $N = 3$, $p = 2$, $q = 4$ and $\alpha \in (0, 3)$, the problem (P_ε) is known in astrophysics as the Gross-Pitaevskii-Poisson equation [4,8,47]. By Theorem 2.3, Theorem 2.4 and Proposition 2.3 we conclude that (P_ε) admits a positive ground state $u_\varepsilon \in H^1(\mathbb{R}^N)$, which is radially symmetric and radially nonincreasing. Moreover, as $\varepsilon \to 0$, we have

$$u_\varepsilon(0) \sim \varepsilon^{\frac{2+\alpha}{4}}, \quad \|\nabla u_\varepsilon\|_2^2 \simeq \frac{3}{4} S_q^2 \varepsilon^{\frac{1}{2}},$$

$$M(\varepsilon) := \|u_\varepsilon\|_2^2 = \varepsilon^{\frac{\alpha-1}{2}} \left(\frac{1+\alpha}{4} S_p^2 - \Theta(\varepsilon^{\frac{\alpha}{2}})\right), \quad E(u_\varepsilon) = \varepsilon^{\frac{1+\alpha}{2}} \left(\frac{1-\alpha}{8} S_p^2 + O(\varepsilon^{\frac{\alpha}{2}})\right).$$

As $\varepsilon \to \infty$, we have

$$u_\varepsilon(0) \sim \varepsilon^{\frac{1}{2}}, \quad \|\nabla u_\varepsilon\|_2^2 \simeq \frac{3-\alpha}{4} S_p^2 \varepsilon^{\frac{1+\alpha}{2}},$$

$$M(\varepsilon) := \|u_\varepsilon\|_2^2 = \varepsilon^{-\frac{1}{2}} \left(\frac{1}{4} S_q^2 + O(\varepsilon^{-\frac{\alpha}{2}})\right), \quad E(u_\varepsilon) = \varepsilon^{\frac{1}{2}} \left(\frac{1}{8} S_q^2 + O(\varepsilon^{-\frac{\alpha}{2}})\right).$$

Therefore, we conclude that

(1) when $\alpha \in (1,3)$ and $c \in (0, \sup_{\varepsilon>0}\sqrt{M(\varepsilon)})$, (1.4) admits two positive normalized solutions u_c^1 and u_c^2 satisfying

$$\|u_c^1\|_\infty \sim \left(\frac{4}{1+\alpha}S_p^{-2}c^2\right)^{\frac{2+\alpha}{2(\alpha-1)}} \to 0, \quad \text{as } c \to 0,$$

$$\|\nabla u_c^1\|_2^2 \simeq \frac{3}{4}S_q^2\left(\frac{4}{1+\alpha}S_p^{-2}c^2\right)^{\frac{1}{\alpha-1}} \to 0, \quad \text{as } c \to 0,$$

$$E(u_c^1) \simeq -\frac{\alpha-1}{8}(\frac{4}{1+\alpha})^{\frac{1+\alpha}{\alpha-1}}S_p^{-\frac{4}{\alpha-1}}c^{\frac{2(1+\alpha)}{\alpha-1}} \to 0^-, \quad \text{as } c \to 0,$$

$$\|u_c^2\|_\infty \sim \frac{1}{4}S_q^2c^{-2} \to +\infty, \quad \text{as } c \to 0,$$

$$\|\nabla u_c^2\|_2^2 \simeq \frac{3-\alpha}{4}S_p^2\left(\frac{1}{16}S_q^4c^{-4}\right)^{\frac{1+\alpha}{2}} \to +\infty, \quad \text{as } c \to 0,$$

$$E(u_c^2) \simeq \frac{1}{32}S_q^4c^{-2} \to +\infty, \quad \text{as } c \to 0.$$

Moreover, as $c \to 0$, the rescaled family

$$w_c^1(x) := \varepsilon_{1c}^{-\frac{2+\alpha}{4}}u_c^1(\varepsilon_{1c}^{-\frac{1}{2}}x), \quad \varepsilon_{1c} \simeq (\frac{4}{1+\alpha})^{\frac{2}{\alpha-1}}S_p^{-\frac{4}{\alpha-1}}c^{\frac{4}{\alpha-1}},$$

converges in $H^1(\mathbb{R}^3)$, up to a subsequence, to a positive solution w_0 of

$$-\Delta w + w = (I_\alpha * |w|^2)w, \quad x \in \mathbb{R}^3, \tag{2.18}$$

if $\alpha = 2$, the positive solution w_0 of (2.18) is unique [30] and $w_c^1 \to w_0$ in $H^1(\mathbb{R}^3)$, and as $c \to 0$, the rescaled family

$$w_c^2(x) := \varepsilon_{2c}^{-\frac{1}{2}}u_c^2(\varepsilon_{2c}^{-\frac{1}{2}}x), \quad \varepsilon_{2c} \simeq \frac{1}{16}S_q^4c^{-4},$$

converges in $H^1(\mathbb{R}^3)$ to the unique positive solution w_∞ of $-\Delta w + w = w^3$.

(2) when $\alpha = 1$ and $c \in (0, \frac{\sqrt{2}}{2}S_p)$, (1.4) admits a positive normalized solution u_c satisfying

$$\|u_c\|_\infty \sim \frac{1}{4}S_q^2c^{-2} \to +\infty, \quad \text{as } c \to 0,$$

$$\|\nabla u_c\|_2^2 \simeq \frac{1}{32}S_p^2S_q^4c^{-4} \to +\infty, \quad \text{as } c \to 0,$$

$$E(u_c) \simeq \frac{1}{32}S_q^4c^{-2} \to +\infty, \quad \text{as } c \to 0,$$

$$\|u_c\|_\infty \sim (\frac{1}{2}S_p^2 - c^2)^{\frac{3}{2}} \to 0, \quad \text{as } c \to \frac{\sqrt{2}}{2}S_p,$$

$$\|\nabla u_c\|_2^2 \sim \frac{3}{4}S_q^2(\frac{1}{2}S_p^2 - c^2) \to 0, \ \text{ as } c \to \frac{\sqrt{2}}{2}S_p,$$
$$E(u_c) = O\left((\frac{1}{2}S_p^2 - c^2)^3\right) \to 0, \ \text{ as } c \to \frac{\sqrt{2}}{2}S_p.$$

Moreover, as $c \to \frac{\sqrt{2}}{2}S_p$, the rescaled family

$$w_c(x) := \varepsilon_c^{-\frac{3}{4}} u_c(\varepsilon_c^{-\frac{1}{2}}x), \quad \varepsilon_c \sim (\frac{1}{2}S_p^2 - c^2)^2,$$

converges in $H^1(\mathbb{R}^3)$, up to a subsequence, to a positive solution w_0 of (2.18) with $\alpha = 1$, and as $c \to 0$, the rescaled family

$$w_c(x) := \varepsilon_c^{-\frac{1}{2}} u_c(\varepsilon_c^{-\frac{1}{2}}x), \quad \varepsilon_c \simeq \frac{1}{16}S_q^4 c^{-4},$$

converges in $H^1(\mathbb{R}^3)$ to the unique positive solution w_∞ of $-\Delta w + w = w^3$.

(3) when $\alpha \in (0, 1)$ and $c > 0$, (1.4) admits a positive normalized solution u_c satisfying

$$\|u_c\|_\infty \sim \frac{1}{4}S_q^2 c^{-2} \to +\infty, \ \text{ as } c \to 0,$$
$$\|\nabla u_c\|_2^2 \simeq \frac{3-\alpha}{4}S_p^2\left(\frac{1}{4}S_q^2 c^{-2}\right)^{1+\alpha} \to +\infty, \ \text{ as } c \to 0,$$
$$E(u_c) \simeq \frac{1}{32}S_q^4 c^{-2} \to +\infty, \ \text{ as } c \to 0,$$
$$\|u_c\|_\infty \sim \left(\frac{4}{1+\alpha}S_p^{-2}c^2\right)^{\frac{2+\alpha}{2(\alpha-1)}} \to 0, \ \text{ as } c \to \infty,$$
$$\|\nabla u_c\|_2^2 \simeq \frac{3}{4}S_q^2\left(\frac{4}{1+\alpha}S_p^{-2}c^2\right)^{\frac{1}{\alpha-1}} \to 0, \ \text{ as } c \to \infty,$$
$$E(u_c) \simeq \frac{1-\alpha}{8}(\frac{4}{1+\alpha})^{\frac{1+\alpha}{\alpha-1}} S_p^{-\frac{4}{\alpha-1}} c^{\frac{2(1+\alpha)}{\alpha-1}} \to 0^+, \ \text{ as } c \to \infty.$$

Moreover, as $c \to \infty$, the rescaled family

$$w_c(x) := \varepsilon_c^{-\frac{2+\alpha}{4}} u_c(\varepsilon_c^{-\frac{1}{2}}x), \quad \varepsilon_c \simeq (\frac{4}{1+\alpha})^{\frac{2}{\alpha-1}} S_p^{-\frac{4}{\alpha-1}} c^{\frac{4}{\alpha-1}},$$

converges in $H^1(\mathbb{R}^3)$, up to a subsequence, to a positive solution w_0 of (2.18), and as $c \to 0$, the rescaled family

$$w_c(x) := \varepsilon_c^{-\frac{1}{2}} u_c(\varepsilon_c^{-\frac{1}{2}}x), \quad \varepsilon_c \simeq \frac{1}{16}S_q^4 c^{-4},$$

converges in $H^1(\mathbb{R}^3)$ to the unique positive solution w_∞ of $-\Delta w + w = w^3$.

3. Preliminaries

In this section, we present some preliminary results which are needed in the proof of our main results. First, we consider the following Choquard type equation with combined nonlinearities:

$$-\Delta u+u=\mu(I_\alpha*|u|^p)|u|^{p-2}u+\lambda|u|^{q-2}u,\quad \text{in } \mathbb{R}^N, \qquad (Q_{\mu,\lambda})$$

where $N\ge 3,\ \alpha\in(0,N),\ p\in[\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}],\ q\in(2,2^*]$, $\mu>0$ and $\lambda>0$ are two parameters.

It has been proved in [25] that any weak solution of $(Q_{\mu,\lambda})$ in $H^1(\mathbb{R}^N)$ has additional regularity properties, which allows us to establish the Pohožaev identity for all finite energy solutions.

Lemma 3.1. *If $u\in H^1(\mathbb{R}^N)$ is a solution of $(Q_{\mu,\lambda})$, then $u\in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $r>1$. Moreover, u satisfies the Pohožaev identity*

$$P_{\mu,\lambda}(u):=\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2+\frac{N}{2}\int_{\mathbb{R}^N}|u|^2-\frac{N+\alpha}{2p}\mu\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p-\frac{N}{q}\lambda\int_{\mathbb{R}^N}|u|^q=0.$$

It is well known that any weak solution of $(Q_{\mu,\lambda})$ corresponds to a critical point of the action functionals $I_{\mu,\lambda}$ defined by

$$I_{\mu,\lambda}(u):=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2+|u|^2-\frac{\mu}{2p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p-\frac{\lambda}{q}\int_{\mathbb{R}^N}|u|^q, \tag{3.1}$$

which is well defined and is of C^1 in $H^1(\mathbb{R}^N)$. A nontrivial solution $u_{\mu,\lambda}\in H^1(\mathbb{R}^N)$ of $(Q_{\mu,\lambda})$ is called a ground state if

$$I_{\mu,\lambda}(u_{\mu,\lambda})=m_{\mu,\lambda}:=\inf\{I_{\mu,\lambda}(u):\ u\in H^1(\mathbb{R}^N)\setminus\{0\}\text{ and }I'_{\mu,\lambda}(u)=0\}. \tag{3.2}$$

In [24,25] (see also the proof of the main results in [25]), it has been shown that

$$m_{\mu,\lambda}=\inf_{u\in\mathcal{M}_{\mu,\lambda}}I_{\mu,\lambda}(u)=\inf_{u\in\mathcal{P}_{\mu,\lambda}}I_{\mu,\lambda}(u), \tag{3.3}$$

where $\mathcal{M}_{\mu,\lambda}$ and $\mathcal{P}_{\mu,\lambda}$ are the corresponding Nehari and Pohožaev manifolds defined by

$$\mathcal{M}_{\mu,\lambda}:=\left\{u\in H^1(\mathbb{R}^N)\setminus\{0\}\ \middle|\ \int_{\mathbb{R}^N}|\nabla u|^2+|u|^2=\mu\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p+\lambda\int_{\mathbb{R}^N}|u|^q\right\}$$

and

$$\mathcal{P}_{\mu,\lambda}:=\left\{u\in H^1(\mathbb{R}^N)\setminus\{0\}\ \middle|\ P_{\mu,\lambda}(u)=0\right\},$$

respectively. Moreover, the following min-max descriptions are valid:

Lemma 3.2. *Let*

$$u_t(x) = \begin{cases} u(\frac{x}{t}), & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases}$$

then

$$m_{\mu,\lambda} = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \sup_{t \geq 0} I_{\mu,\lambda}(tu) = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \sup_{t \geq 0} I_{\mu,\lambda}(u_t). \tag{3.4}$$

In particular, we have $m_{\mu,\lambda} = I_{\mu,\lambda}(u_{\mu,\lambda}) = \sup_{t>0} I_{\mu,\lambda}(tu_{\mu,\lambda}) = \sup_{t>0} I_{\mu,\lambda}((u_{\mu,\lambda})_t)$.

When $\mu = 1$ and $\lambda = 0$, then the equation $(Q_{\mu,\lambda})$ reduces to

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{$Q_{1,0}$}$$

when $\mu = 0$ and $\lambda = 1$, then the equation $(Q_{\mu,\lambda})$ reduces to

$$\Delta u + u = |u|^{q-2}u, \quad \text{in } \mathbb{R}^N. \tag{$Q_{0,1}$}$$

Then the corresponding Nehari manifolds are as follows.

$$\mathcal{M}_{1,0} = \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \right\}.$$

$$\mathcal{M}_{0,1} = \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 = \int_{\mathbb{R}^N} |u|^q \right\}.$$

It is known that

$$m_{1,\lambda} = \inf_{u \in \mathcal{M}_{1,\lambda}} I_{1,\lambda}(u), \qquad m_{1,0} := \inf_{u \in \mathcal{M}_{1,0}} I_{1,0}(u), \tag{3.5}$$

and

$$m_{\mu,1} := \inf_{u \in \mathcal{M}_{\mu,1}} I_{\mu,1}(u), \qquad m_{0,1} := \inf_{u \in \mathcal{M}_{0,1}} I_{0,1}(u) \tag{3.6}$$

are well-defined and positive.

Let $u_{\mu,\lambda}$ be the ground state for $(Q_{\mu,\lambda})$, then we have the following.

Lemma 3.3. *The solution sequences* $\{u_{1,\lambda}\}$ *and* $\{u_{\mu,1}\}$ *are bounded in* $H^1(\mathbb{R}^N)$.

Proof. It is not hard to see that $m_{1,\lambda} \le m_{1,0} \le C < +\infty$. If $q \ge 2p$, then

$$\begin{aligned} m_{1,\lambda} &= I_{1,\lambda}(u_{1,\lambda}) = I_{1,\lambda}(u_{1,\lambda}) - \frac{1}{2p} I'_{1,\lambda}(u_{1,\lambda})u_{1,\lambda} \\ &= \left(\frac{1}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} |\nabla u_{1,\lambda}|^2 + |u_{1,\lambda}|^2 + \left(\frac{1}{2p} - \frac{1}{q}\right) \lambda \int_{\mathbb{R}^N} |u_{1,\lambda}|^q \\ &\ge \left(\frac{1}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} |\nabla u_{1,\lambda}|^2 + |u_{1,\lambda}|^2, \end{aligned}$$

and if $q < 2p$, then

$$\begin{aligned} m_{1,\lambda} &= I_{1,\lambda}(u_{1,\lambda}) = I_{1,\lambda}(u_{1,\lambda}) - \frac{1}{q} I'_{1,\lambda}(u_{1,\lambda})u_{1,\lambda} \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) \int_{\mathbb{R}^N} |\nabla u_{1,\lambda}|^2 + |u_{1,\lambda}|^2 + \left(\frac{1}{q} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} (I_\lambda * |u_{1,\lambda}|^p)|u_{1,\lambda}|^p \\ &\ge \left(\frac{1}{2} - \frac{1}{q}\right) \int_{\mathbb{R}^N} |\nabla u_{1,\lambda}|^2 + |u_{1,\lambda}|^2. \end{aligned}$$

Therefore, we conclude that $\{u_{1,\lambda}\}$ is bounded in $H^1(\mathbb{R}^N)$.

Arguing as above, we show that $\{u_{\mu,1}\}$ is bounded in $H^1(\mathbb{R}^N)$. The proof is completed. □

The following well known Hardy-Littlewood-Sobolev inequality can be found in [31].

Lemma 3.4. *Let $p, r > 1$ and $0 < \alpha < N$ with $1/p + (N-\alpha)/N + 1/r = 2$. Let $u \in L^p(\mathbb{R}^N)$ and $v \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(N, \alpha, p)$, independent of u and v, such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)v(y)}{|x-y|^{N-\alpha}} \right| \le C(N, \alpha, p) \|u\|_p \|v\|_r.$$

If $p = r = \frac{2N}{N+\alpha}$, then

$$C(N, \alpha, p) = C_\alpha(N) = \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right\}^{-\frac{\alpha}{N}}.$$

Remark 3.1. By the Hardy-Littlewood-Sobolev inequality, for any $v \in L^s(\mathbb{R}^N)$ with $s \in (1, \frac{N}{\alpha})$, $I_\alpha * v \in L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)$ and

$$\|I_\alpha * v\|_{\frac{Ns}{N-\alpha s}} \le A_\alpha(N) C(N, \alpha, s) \|v\|_s. \tag{3.7}$$

Lemma 3.5. (P. L. Lions [32]) *Let $r > 0$ and $2 \le q \le 2^*$. If (u_n) is bounded in $H^1(\mathbb{R}^N)$ and if*

$$\sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^q \to 0, \ \text{as } n \to \infty,$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Moreover, if $q = 2^*$, then $u_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$.

Lemma 3.6. *Let $r>0$, $N\geq 3$, $\alpha\in(0,N)$ and $\frac{N+\alpha}{N}\leq p\leq\frac{N+\alpha}{N-2}$. If (u_n) be bounded in $H^1(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_r(z)}\int_{B_r(z)}\frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}dxdy=0,$$

then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^s dx=\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha*|u_n|^t)|u_n|^t dx=0,$$

for any $2<s<2^*$ and $\frac{N+\alpha}{N}<t<\frac{N+\alpha}{N-2}$. Moreover, if $p=\frac{N+\alpha}{N-2}$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}dx=\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{N+\alpha}{N-2}})|u_n|^{\frac{N+\alpha}{N-2}}dx=0.$$

Proof. Similar to the proof of [6, Lemma 3.8], for any $p\in[\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}]$, it is easy to show that

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_r(z)}\int_{B_r(z)}\frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}dxdy=0$$

is equivalent to the following condition

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_r(z)}|u_n|^{\frac{2Np}{N+\alpha}}dx=0.$$

Then the conclusion follows from Lemma 3.5. The proof is complete. □

Lemma 3.7. (Radial Lemma A.II, H. Berestycki and P. L. Lions [3]) *Let $N\geq 2$, then every radial function $u\in H^1(\mathbb{R}^N)$ is almost everywhere equal to a function $\tilde{u}(x)$, continuous for $x\neq 0$, such that*

$$|\tilde{u}(x)|\leq C_N|x|^{(1-N)/2}\|u\|_{H^1(\mathbb{R}^N)}\qquad \textit{for } |x|\geq\alpha_N, \tag{3.8}$$

where C_N and α_N depend only on the dimension N.

Lemma 3.8. (Radial Lemma A.III. H. Berestycki and P. L. Lions [3]) *Let $N\geq 3$, then every radial function u in $D^{1,2}(\mathbb{R}^N)$ is almost everywhere equal to a function $\tilde{u}(x)$, continuous for $x\neq 0$, such that*

$$|\tilde{u}(x)|\leq C_N|x|^{(2-N)/2}\|u\|_{D^{1,2}(\mathbb{R}^N)}\qquad \textit{for } |x|\geq 1, \tag{3.9}$$

where C_N only depends on N.

Lemma 3.9. *Let* $0 < \alpha < N$ *and* $0 \le f \in L^1(\mathbb{R}^N)$. *Assume that*

$$\lim_{|x|\to\infty} \frac{\int_{|y|\le|x|} f(y)|y|dy}{|x|} = 0, \tag{3.10}$$

$$\lim_{|x|\to\infty} \int\limits_{|y-x|\le|x|/2} \frac{f(y)dy}{|x-y|^{N-\alpha}} = 0. \tag{3.11}$$

Then as $|x| \to \infty$,

$$\int\limits_{\mathbb{R}^N} \frac{f(y)dy}{|x-y|^{N-\alpha}} = \frac{\|f\|_{L^1}}{|x|^{N-\alpha}} + o\left(\frac{1}{|x|^{N-\alpha}}\right). \tag{3.12}$$

Note that $f \in L^1(\mathbb{R}^N)$ alone is not sufficient to obtain (3.12) even if f is radially symmetric, see [42].

Proof. Fix $0 \ne x \in \mathbb{R}^N$, we decompose $\mathbb{R}^N$ as the union of three sets $B = \{y : |y-x| < |x|/2\}$, $A = \{y \notin B : |y| \le |x|\}$ and $C = \{y \notin B : |y| > |x|\}$.

We want to estimate the quantity

$$\left|\int\limits_{A\cup C} f(y)\left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x|^{N-\alpha}}\right)dy\right| \le \int\limits_{A\cup C} f(y)\left|\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x|^{N-\alpha}}\right|dy.$$

Since $|x|/2 \le |x-y| \le 2|x|$ for all $y \in A$, by the Mean Value Theorem we have

$$\left|\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x|^{N-\alpha}}\right| \le \frac{c_1|y|}{|x|^{N-\alpha+1}}, \ (y \in A),$$

where $c_1 = (N-\alpha)2^{N-\alpha+1}$. Thus

$$\left|\int\limits_{A} f(y)\left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x|^{N-\alpha}}\right)dy\right| \le \frac{c_1}{|x|^{N-\alpha+1}} \int\limits_{A} f(y)|y|dy.$$

On the other hand, since $|x-y| > |x|/2$ for all $y \in C$, then

$$\left|\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x|^{N-\alpha}}\right| \le \frac{1}{|x|^{N-\alpha}}, \ (y \in C),$$

from which we compute that

$$\left|\int\limits_{C} f(y)\left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x|^{N-\alpha}}\right)dy\right| \le \frac{1}{|x|^{N-\alpha}} \int\limits_{A\cup C} f(y)dy.$$

Then

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}}dy - \frac{\|f\|_{L^1}}{|x|^{N-\alpha}}\right| \\ &\le \frac{c_1}{|x|^{N-\alpha+1}}\int_A f(y)|y|dy + \int_B \frac{f(y)}{|x-y|^{N-\alpha}}dy + \frac{1}{|x|^{N-\alpha}}\int_{B\cup C} f(y)dy.\end{aligned}$$

The conclusion follows from (3.10), (3.11) and since $f \in L^1(\mathbb{R}^N)$. □

Lemma 3.10. *Let $0 < \alpha < N$, $0 \le f(x) \in L^1(\mathbb{R}^N)$ be a radially symmetric function such that*

$$\lim_{|x|\to+\infty} f(|x|)|x|^N = 0. \tag{3.13}$$

If $\alpha \le 1$, we additionally assume that f is monotone non-increasing. Then as $|x| \to +\infty$, we have

$$\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}}dy = \frac{\|f\|_{L^1}}{|x|^{N-\alpha}} + o\left(\frac{1}{|x|^{N-\alpha}}\right). \tag{3.14}$$

Proof. Using (3.13) by l'Hospital rule we conclude that

$$\int_{|y|\le|x|} f(y)|y|dy = \int_0^{|x|} f(r)r^N dr = o(|x|),\ (|x|\to\infty),$$

so (3.10) holds.

For $|x| \gg 1$, using radial estimates on the Riesz kernels in [13, Lemma 2.2] and (3.13) we obtain for $\alpha > 1$:

$$\int_{|y-x|\le|x|/2} \frac{f(y)dy}{|x-y|^{N-\alpha}} \lesssim |x|^{\alpha-1}\int_{|x|/2}^{3|x|/2} f(r)dr = o(|x|^{-(N-\alpha)});$$

for $\alpha = 1$, additionally using monotonicity of f:

$$\begin{aligned}\int_{|y-x|\le|x|/2} \frac{f(y)dy}{|x-y|^{N-\alpha}} &\lesssim \int_{|x|/2}^{3|x|/2} f(r)\log\frac{1}{1-r/|x|}dr \\ &\le f(|x|/2)\int_{|x|/2}^{3|x|/2}\log\frac{1}{1-r/|x|}dr = o(|x|^{-(N-1)});\end{aligned}$$

for $\alpha < 1$, additionally using monotonicity of f:

$$\begin{aligned}\int_{|y-x|\le|x|/2} \frac{f(y)dy}{|x-y|^{N-\alpha}} &\lesssim \int_{|x|/2}^{3|x|/2} \frac{f(r)}{|r-|x||^{1-\alpha}}dr \\ &\le f(|x|/2)\int_{|x|/2}^{3|x|/2}\frac{1}{|1-|x||^{1-\alpha}}dr = o(|x|^{-(N-\alpha)});\end{aligned}$$

so (3.12) holds. This completes the proof. □

The following Moser iteration lemma is given in [1, Proposition B.1]. See also [33] and [14].

Lemma 3.11. *Assume $N \geq 3$. Let $a(x)$ and $b(x)$ be functions on B_4, and let $u \in H^1(B_4)$ be a weak solution to*

$$-\Delta u + a(x)u = b(x)u \qquad in \ \ B_4. \tag{3.15}$$

Suppose that $a(x)$ and u satisfy that

$$a(x) \geq 0 \quad for\ a.\ e.\ x \in B_4, \tag{3.16}$$

and

$$\int_{B_4} a(x)|u(x)v(x)|dx < \infty \quad for\ each\ v \in H_0^1(B_4). \tag{3.17}$$

(i) Assume that for any $\varepsilon \in (0, 1)$, there exists $t_\varepsilon > 0$ such that

$$\|\chi_{[|b|>t_\varepsilon]} b\|_{L^{N/2}(B_4)} \leq \varepsilon,$$

where $[|b| > t] := \{x \in B_4 : \ |b(x)| > t\}$, and $\chi_A(x)$ denotes the characteristic function of $A \subset \mathbb{R}^N$. Then for any $r \in (0, \infty)$, there exists a constant $C(N, r, t_\varepsilon)$ such that

$$\||u|^{r+1}\|_{H^1(B_1)} \leq C(N, r, t_\varepsilon)\|u\|_{L^{2^*}(B_4)}.$$

(ii) Let $s > N/2$ and assume that $b \in L^s(B_4)$. Then there exists a constant $C(N, s, \|b\|_{L^s(B_4)})$ such that

$$\|u\|_{L^\infty(B_1)} \leq C(N, s, \|b\|_{L^s(B_4)})\|u\|_{L^{2^*}(B_4)}.$$

Here, the constants $C(N, r, t_\varepsilon)$ and $C(N, s, \|b\|_{L^s(B_4)})$ in (i) and (ii) remain bounded as long as r, t_ε and $\|b\|_{L^s(B_4)}$ are bounded.

4. Proof of Theorem 2.1

In this section, we always assume that $p = \frac{N+\alpha}{N}$, $q \in (2, 2 + \frac{4}{N})$ and $\lambda > 0$ is a small parameter. It is easy to see that under the rescaling

$$w(x) = \lambda^{-\frac{N}{4-N(q-2)}} v(\lambda^{-\frac{2}{4-N(q-2)}} x), \tag{4.1}$$

the equation (Q_λ) is reduced to

$$-\lambda^\sigma \Delta w + w = (I_\alpha * |w|^p)|w|^{p-2}w + \lambda^\sigma |w|^{q-2}w, \tag{$\bar{Q}_\lambda$}$$

where $\sigma := \frac{4}{4-N(q-2)} > 1$. The corresponding functional is given by

$$J_\lambda(w) := \frac{1}{2}\int_{\mathbb{R}^N} \lambda^\sigma |\nabla w|^2 + |w|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p - \frac{1}{q}\lambda^\sigma \int_{\mathbb{R}^N} |w|^q.$$

Lemma 4.1. *Let $\lambda > 0$, $v \in H^1(\mathbb{R}^N)$ and w be the rescaling (4.1) of v. Then*
(1) $\|w\|_2^2 = \|v\|_2^2, \quad \int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p = \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p,$
(2) $\lambda^\sigma \|\nabla w\|_2^2 = \|\nabla v\|_2^2, \quad \lambda^\sigma \|w\|_q^q = \lambda \|v\|_q^q,$
(3) $I_\lambda(v) = J_\lambda(w)$.

We define the Nehari manifolds as follows.

$$\mathcal{N}_\lambda = \left\{ w \in H^1(\mathbb{R}^N) \setminus \{0\} \;\middle|\; \lambda^\sigma \int\limits_{\mathbb{R}^N} |\nabla w|^2 + \int\limits_{\mathbb{R}^N} |w|^2 = \int\limits_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p + \lambda^\sigma \int\limits_{\mathbb{R}^N} |w|^q \right\}$$

and

$$\mathcal{N}_0 = \left\{ w \in H^1(\mathbb{R}^N) \setminus \{0\} \;\middle|\; \int\limits_{\mathbb{R}^N} |w|^2 = \int\limits_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p \right\}.$$

Then

$$m_\lambda := \inf_{w \in \mathcal{N}_\lambda} J_\lambda(w), \quad \text{and} \quad m_0 := \inf_{u \in \mathcal{N}_0} J_0(u)$$

are well-defined and positive. Moreover, J_0 is attained on $\mathcal{N}_0$ and

$$m_0 := \inf_{w \in \mathcal{N}_0} J_0(w) = \frac{\alpha}{2(N+\alpha)} S_1^{\frac{N+\alpha}{\alpha}}.$$

For $w \in H^1(\mathbb{R}^N) \setminus \{0\}$, we set

$$\tau_1(w) = \frac{\int_{\mathbb{R}^N} |w|^2}{\int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p}. \tag{4.2}$$

Then $(\tau_1(w))^{\frac{N}{2\alpha}} w \in \mathcal{N}_0$ for any $w \in H^1(\mathbb{R}^N) \setminus \{0\}$, and $w \in \mathcal{N}_0$ if and only if $\tau_1(w) = 1$.

Define the Pohožaev manifold as follows

$$\mathcal{P}_\lambda := \{ w \in H^1(\mathbb{R}^N) \setminus \{0\} \mid P_\lambda(w) = 0 \},$$

where

$$\begin{aligned} P_\lambda(w) : \quad = \quad & \tfrac{\lambda^\sigma (N-2)}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + \tfrac{N}{2} \int_{\mathbb{R}^N} |w|^2 \\ & - \tfrac{N+\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p - \tfrac{\lambda^\sigma N}{q} \int_{\mathbb{R}^N} |w|^q. \end{aligned} \tag{4.3}$$

Let $v_\lambda \in H^1(\mathbb{R}^N)$ be the ground state for (Q_λ) and

$$w_\lambda(x) = \lambda^{-\frac{N}{4-N(q-2)}} v_\lambda(\lambda^{-\frac{2}{4-N(q-2)}} x).$$

Then by Lemma 3.1, $w_\lambda \in \mathcal{P}_\lambda$. Moreover, we have the following minimax characterizations for the least energy level m_λ.

$$m_\lambda = \inf_{w\in H^1(\mathbb{R}^N)\setminus\{0\}} \sup_{t\geq 0} J_\lambda(tw) = \inf_{w\in H^1(\mathbb{R}^N)\setminus\{0\}} \sup_{t\geq 0} J_\lambda(w_t). \tag{4.4}$$

In particular, we have $m_\lambda = J_\lambda(w_\lambda) = \sup_{t>0} J_\lambda(tw_\lambda) = \sup_{t>0} J_\lambda((w_\lambda)_t)$. A similar result also holds for m_0 and J_0.

Lemma 4.2. *The rescaled family of solutions $\{w_\lambda\}$ is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Since $\{w_\lambda\}$ is bounded in $L^2(\mathbb{R}^N)$, it suffices to show that it is also bounded in $D^{1,2}(\mathbb{R}^N)$. By $w_\lambda \in \mathcal{N}_\lambda \cap \mathcal{P}_\lambda$, we obtain

$$\lambda^\sigma \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \int_{\mathbb{R}^N} |w_\lambda|^2 = \int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^p)|w_\lambda|^p + \lambda^\sigma \int_{\mathbb{R}^N} |w_\lambda|^q,$$

$$\frac{\lambda^\sigma(N-2)}{2} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \frac{N}{2} \int_{\mathbb{R}^N} |w_\lambda|^2 = \frac{N+\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^p)|w_\lambda|^p + \frac{\lambda^\sigma N}{q} \int_{\mathbb{R}^N} |w_\lambda|^q.$$

Therefore, we have

$$\|\nabla v_\lambda\|_2^2 = \lambda^\sigma \|\nabla w_\lambda\|_2^2 = \frac{N(q-2)}{2q} \lambda^\sigma \int_{\mathbb{R}^N} |w_\lambda|^q = \frac{N(q-2)}{2q} \lambda \int_{\mathbb{R}_N} |v_\lambda|^q.$$

Particularly, we have

$$\|\nabla w_\lambda\|_2^2 = \frac{N(q-2)}{2q} \|w_\lambda\|_q^q. \tag{4.5}$$

By the Gagliardo-Nirenberg Inequality, we obtain

$$\|v_\lambda\|_q^q \leq C \|\nabla v_\lambda\|_2^{\frac{N(q-2)}{2}} \|v_\lambda\|_2^{\frac{2N-q(N-2)}{2}}.$$

Therefore, we get

$$\|\nabla v_\lambda\|_2^{\frac{4-N(q-2)}{2}} \leq C \frac{N(q-2)}{2q} \lambda \|v_\lambda\|_2^{\frac{2N-q(N-2)}{2}}.$$

Hence,

$$\lambda^\sigma \|\nabla w_\lambda\|_2^2 = \|\nabla v_\lambda\|_2^2 \leq \tilde{C} \lambda^\sigma \|v_\lambda\|_2^{\frac{2[2N-q(N-2)]}{4-N(q-2)}},$$

which together with the boundedness of $\|v_\lambda\|_2$ implies that w_λ is bounded in $D^{1,2}(\mathbb{R}^N)$. □

Now, we give the estimates on $\tau_1(w_\lambda)$ and the least energy m_λ.

Lemma 4.3. *$1 < \tau_1(w_\lambda) \le 1 + O(\lambda)$ and $m_\lambda = m_0 + O(\lambda)$ as $\lambda \to 0$.*

Proof. First, since $w_\lambda \in \mathcal{N}_\lambda$, by (4.5), it follows that

$$\tau_1(w_\lambda) = \frac{\int_{\mathbb{R}^N} |w_\lambda|^2}{\int_{\mathbb{R}^N}(I_\alpha * |w_\lambda|^p)|w_\lambda|^p} > \frac{\int_{\mathbb{R}^N} \lambda^\sigma |\nabla w_\lambda|^2 + |w_\lambda|^2 - \lambda^\sigma \int_{\mathbb{R}^N} |w_\lambda|^q}{\int_{\mathbb{R}^N}(I_\alpha * |w_\lambda|^p)|w_\lambda|^p} = 1 \tag{4.6}$$

and by Lemma 4.1 and the Sobolev inequality, we have

$$\begin{aligned} \tau_1(w_\lambda) &= \frac{\int_{\mathbb{R}^N} |w_\lambda|^2}{\int_{\mathbb{R}^N}(I_\alpha*|w_\lambda|^p)|w_\lambda|^p} \\ &\le \frac{\int_{\mathbb{R}^N} \lambda^\sigma |\nabla w_\lambda|^2+|w_\lambda|^2}{\int_{\mathbb{R}^N}(I_\alpha*|w_\lambda|^p)|w_\lambda|^p+\lambda^\sigma \int_{\mathbb{R}^N} |w_\lambda|^q-\lambda^\sigma \int_{\mathbb{R}^N} |w_\lambda|^q} \\ &= \frac{\|v_\lambda\|^2}{\|v_\lambda\|^2-\lambda\|v_\lambda\|_q^q} \le \frac{1}{1-\lambda C\|v_\lambda\|^{q-2}}. \end{aligned} \tag{4.7}$$

Since $\|v_\lambda\|$ is bounded, it follows that $1 < \tau_1(w_\lambda) \le 1 + O(\lambda)$ as $\lambda \to 0$.

For $w \in H^1(\mathbb{R}^N)$, let

$$w_t(x) = \begin{cases} w(x/t) & t > 0, \\ 0, & t = 0. \end{cases}$$

Then by Lemma 3.2 and Pohožaev's identity, it is easy to show that $\sup_{t\ge 0} J_\lambda((w_\lambda)_t) = J_\lambda(w_\lambda) = m_\lambda$. Therefore, we get

$$\begin{aligned} m_0 &\le \sup_{t\ge 0} J_0((w_\lambda)_t) = J_0((w_\lambda)_t)|_{t=\tau_1((w_\lambda))^{1/\alpha}} \\ &\le \sup_{t\ge 0} J_\lambda((w_\lambda)_t) + \lambda^\sigma(\tau_1(w_\lambda))^{N/\alpha} \int_{\mathbb{R}^N} |w_\lambda|^q \\ &\le m_\lambda + \lambda(1+O(\lambda))^{N/\alpha}\|v_\lambda\|_q^q \\ &= m_\lambda + O(\lambda). \end{aligned}$$

On the other hand, let $U \in \mathcal{N} \subset H^1(\mathbb{R}^N)$ be such that

$$S_1 = \frac{\int_{\mathbb{R}^N} |U|^2}{\left(\int_{\mathbb{R}^N}(I_\alpha * |U|^p)|U|^p\right)^{1/p}}.$$

Then $\int_{\mathbb{R}^N} |U|^2 = \int_{\mathbb{R}^N}(I_\alpha * |U|^{\frac{N+\alpha}{N}})|U|^{\frac{N+\alpha}{N}} = S_1^{\frac{N+\alpha}{\alpha}}$ and $m_0 = J_0(U) = \frac{\alpha}{2(N+\alpha)} S_1^{\frac{N+\alpha}{\alpha}}$. By Lemma 3.2 again, we obtain

$$\begin{aligned} m_\lambda &\le \sup_{t\ge 0} J_\lambda(tU) \\ &= \sup_{t\ge 0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^N} \lambda^\sigma|\nabla U|^2 + |U|^2 - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N}(I_\alpha * |U|^p)|U|^p - \frac{\lambda^\sigma t^q}{q}\int_{\mathbb{R}^N} |U|^q\right\} \\ &\le \sup_{t\ge 0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^N} |U|^2 - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N}(I_\alpha * |U|^p)|U|^p\right\} \\ &\quad + \lambda^\sigma \sup_{t\ge 0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla U|^2 - \frac{t^q}{q}\int_{\mathbb{R}^N} |U|^q\right\} \\ &= J_0(U) + O(\lambda^\sigma) \\ &= m_0 + o(\lambda). \end{aligned}$$

The proof is completed. □

Lemma 4.4. $\|w_\lambda\|_2^2 \sim 1$ *as* $\lambda \to 0$.

Proof. By the definition of $\tau_1(w_\lambda)$, Lemma 4.3 and the Hardy-Littlewood-Sobolev inequality, for small $\lambda > 0$, we have

$$\|w_\lambda\|_2^2 = \tau_1(w_\lambda) \int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^p)|w_\lambda|^p \le 2S_1^{-p}\|w_\lambda\|_2^{2p},$$

and thus it follows that

$$\|w_\lambda\|_2^2 \ge 2^{-\frac{N}{\alpha}} S_1^{\frac{N+\alpha}{\alpha}},$$

which together with the boundedness of w_λ implies that $\|w_\lambda\|_2^2 \sim 1$ as $\lambda \to 0$. The proof is completed. □

Now, we give the following estimates on the least energy.

Lemma 4.5. *Let* $N \ge 3$ *and* $q \in (2, 2 + \frac{4}{N})$*, then*

$$m_0 - m_\lambda \sim \lambda^\sigma \qquad as \quad \lambda \to 0.$$

Proof. By Lemma 3.2, Lemma 4.3 and the boundedness of $\{w_\lambda\}$, we find

$$\begin{aligned} m_0 &\le \sup_{t\ge 0} J_0((w_\lambda)_t) = J_0((w_\lambda)_{t_\lambda}) \\ &\le \sup_{t\ge 0} J_\lambda((w_\lambda)_t) + \lambda^\sigma \left(\frac{t_\lambda^N}{q}\int_{\mathbb{R}^N} |w_\lambda|^q - \frac{t_\lambda^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla w_\lambda|^2\right) \\ &\le m_\lambda + C\lambda^\sigma, \end{aligned} \tag{4.8}$$

where

$$t_\lambda = \left(\frac{\int_{\mathbb{R}^N} |w_\lambda|^2}{\int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^p)|w_\lambda|^p}\right)^{\frac{1}{\alpha}} = (\tau_1(w_\lambda))^{\frac{1}{\alpha}}.$$

For each $\rho > 0$, the family $U_\rho(x) := \rho^{-\frac{N}{2}} U_1(x/\rho)$ are radial ground states of $v = (I_\alpha * |v|^p)v^{p-1}$, and verify that

$$\|\nabla U_\rho\|_2^2 = \rho^{-2}\|\nabla U_1\|_2^2, \qquad \int_{\mathbb{R}^N} |U_\rho|^q = \rho^{N-\frac{N}{2}q} \int_{\mathbb{R}^N} |U_1|^q. \tag{4.9}$$

Let $g_0(\rho) = \frac{1}{q}\int_{\mathbb{R}^N} |U_\rho|^q - \frac{1}{2}\int_{\mathbb{R}^N} |\nabla U_\rho|^2$. Then there exists $\rho_0 = \rho(q) \in (0, +\infty)$ with

$$\rho_0 = \left(\frac{2q\int_{\mathbb{R}^N} |\nabla U_1|^2}{N(q-2)\int_{\mathbb{R}^N} |U_1|^q}\right)^{\frac{2}{4-N(q-2)}}$$

such that

$$g_0(\rho_0) = \sup_{\rho>0} g_0(\rho) = \frac{4-N(q-2)}{2N(q-2)} \left(\frac{N(q-2)\int_{\mathbb{R}^N} |U_1|^q}{2q\int_{\mathbb{R}^N} |\nabla U_1|^2} \right)^{\frac{4}{4-N(q-2)}} \int_{\mathbb{R}^N} |\nabla U_1|^2.$$

Let $U_0 = U_{\rho_0}$, then there exists $t_\lambda \in (0, +\infty)$ such that

$$\begin{aligned} m_\lambda &\le \sup_{t\ge 0} J_\lambda(tU_0) = J_\lambda(t_\lambda U_0) \\ &= \frac{t_\lambda^2}{2}\int_{\mathbb{R}^N} |U_0|^2 - \frac{t_\lambda^{2p}}{2p}\int_{\mathbb{R}^N} (I_\alpha * |U_0|^p)|U_0|^p - \lambda^\sigma \{\frac{t_\lambda^q}{q}\int_{\mathbb{R}^N} |U_0|^q - \frac{t_\lambda^2}{2}\int_{\mathbb{R}^N} |\nabla U_0|^2\} \\ &\le \sup_{t\ge 0}\left(\frac{t^2}{2} - \frac{t^{2p}}{2p}\right)\int_{\mathbb{R}^N} |U_0|^2 - \lambda^\sigma \{\frac{t_\lambda^q}{q}\int_{\mathbb{R}^N} |U_0|^q - \frac{t_\lambda^2}{2}\int_{\mathbb{R}^N} |\nabla U_0|^2\} \\ &= m_0 - \lambda^\sigma \{\frac{t_\lambda^q}{q}\int_{\mathbb{R}^N} |U_0|^q - \frac{t_\lambda^2}{2}\int_{\mathbb{R}^N} |\nabla U_0|^2\}. \end{aligned} \tag{4.10}$$

If $t_\lambda \ge 1$, then

$$\int_{\mathbb{R}^N} |U_0|^2 + \lambda^\sigma \int_{\mathbb{R}^N} |\nabla U_0|^2 \ge t_\lambda^{\min\{\frac{2\alpha}{N}, q-2\}} \left\{ \int_{\mathbb{R}^N} (I_\alpha * |U_0|^p)|U_0|^p + \lambda^\sigma \int_{\mathbb{R}^N} |U_0|^q \right\}.$$

Hence

$$t_\lambda \le \left(\frac{\int_{\mathbb{R}^N} |U_0|^2 + \lambda^\sigma \int_{\mathbb{R}^N} |\nabla U_0|^2}{\int_{\mathbb{R}^N} (I_\alpha * |U_0|^p)|U_0|^p + \lambda^\sigma \int_{\mathbb{R}^N} |U_0|^q} \right)^{\frac{1}{\min\{\frac{2\alpha}{N}, q-2\}}}.$$

If $t_\lambda \le 1$, then

$$\int_{\mathbb{R}^N} |U_0|^2 + \lambda^\sigma \int_{\mathbb{R}^N} |\nabla U_0|^2 \le t_\lambda^{\min\{\frac{2\alpha}{N}, q-2\}} \left\{ \int_{\mathbb{R}^N} (I_\alpha * |U_0|^p)|U_0|^p + \lambda^\sigma \int_{\mathbb{R}^N} |U_0|^q \right\}.$$

Hence

$$t_\lambda \ge \left(\frac{\int_{\mathbb{R}^N} |U_0|^2 + \lambda^\sigma \int_{\mathbb{R}^N} |\nabla U_0|^2}{\int_{\mathbb{R}^N} (I_\alpha * |U_0|^p)|U_0|^p + \lambda^\sigma \int_{\mathbb{R}^N} |U_0|^q} \right)^{\frac{1}{\min\{\frac{2\alpha}{N}, q-2\}}}.$$

Since

$$\int_{\mathbb{R}^N} (I_\alpha * |U_0|^p)|U_0|^p = \int_{\mathbb{R}^N} |U_0|^2 \qquad \text{and} \qquad \int_{\mathbb{R}^N} |U_0|^q > \int_{\mathbb{R}^N} |\nabla U_0|^2,$$

we conclude that

$$\left(\frac{\int_{\mathbb{R}^N} |U_0|^2 + \lambda^\sigma \int_{\mathbb{R}^N} |\nabla U_0|^2}{\int_{\mathbb{R}^N} (I_\alpha * |U_0|^p)|U_0|^p + \lambda^\sigma \int_{\mathbb{R}^N} |U_0|^q} \right)^{\frac{1}{\min\{\frac{2\alpha}{N}, q-2\}}} \le t_\lambda \le 1. \tag{4.11}$$

Therefore, $\lim_{\lambda\to 0} t_\lambda = 1$ and hence there exists a constant $C > 0$ such that

$$m_\lambda \le m_0 - C\lambda^\sigma,$$

for small $\lambda > 0$. The proof is complete. □

Denote

$$\mathbb{D}(u) := \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}}.$$

The following result is a special case of the classical Brezis-Lieb lemma [5] for Riesz potentials, for a proof, we refer the reader to [37, Lemma 2.4].

Lemma 4.6. *Let $N \in \mathbb{N}$, $\alpha \in (0, N)$, and $(w_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^N)$. If $w_n \to w$ almost everywhere on $\mathbb{R}^N$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \mathbb{D}(w_n) - \mathbb{D}(w_n - w_0) = \mathbb{D}(w_0).$$

Lemma 4.7. $\|w_\lambda\|_2^2 \sim \mathbb{D}(w_\lambda) \sim \|\nabla w_\lambda\|_2^2 \sim \|w_\lambda\|_q^q \sim 1$ *as* $\lambda \to 0$.

Proof. It follows from Lemma 4.3 that

$$\begin{aligned} m_0 &\le J_0((\tau_1(w_\lambda))^{\frac{N}{2\alpha}} w_\lambda) \\ &= \tfrac{1}{2}(\tau_1(w_\lambda))^{\frac{N}{\alpha}}\|w_\lambda\|_2^2 - \tfrac{1}{2p}(\tau_1(w_\lambda))^{\frac{N+\alpha}{\alpha}} \textstyle\int_{\mathbb{R}^N}(I_\alpha * |w_\lambda|^p)|w_\lambda|^p \\ &\le (\tau_1(w_\lambda))^{\frac{N}{\alpha}}\left[\tfrac{1}{2}\|w_\lambda\|_2^2 - \tfrac{1}{2p}\textstyle\int_{\mathbb{R}^N}(I_\alpha * |w_\lambda|^p)|w_\lambda|^p\right]. \end{aligned} \tag{4.12}$$

Since $w_\lambda \in \mathcal{N}_\lambda$, by Lemma 4.4, we obtain

$$\begin{aligned} \tau_1(w_\lambda) &= \frac{\int_{\mathbb{R}^N}|w_\lambda|^2}{\int_{\mathbb{R}^N}(I_\alpha*|w_\lambda|^p)|w_\lambda|^p} \\ &= \frac{\int_{\mathbb{R}^N}|w_\lambda|^2}{\int_{\mathbb{R}^N}|w_\lambda|^2+\lambda^\sigma(\int_{\mathbb{R}^N}|\nabla w_\lambda|^2-\int_{\mathbb{R}^N}|w_\lambda|^q)} \\ &= \frac{\int_{\mathbb{R}^N}|w_\lambda|^2}{\int_{\mathbb{R}^N}|w_\lambda|^2-\lambda^\sigma\frac{2N-q(N-2)}{2q}\int_{\mathbb{R}^N}|w_\lambda|^q} \\ &\le 1 + C_1\lambda^\sigma\|w_\lambda\|_q^q. \end{aligned} \tag{4.13}$$

Therefore, by (4.5), (4.12) and (4.13), we obtain

$$\begin{aligned} m_\lambda &= \tfrac{1}{2}\textstyle\int_{\mathbb{R}^N}\lambda^\sigma|\nabla w_\lambda|_2^2 + |w_\lambda|_2^2 - \tfrac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|w_\lambda|^p)|w_\lambda|^p - \tfrac{\lambda^\sigma}{q}\int_{\mathbb{R}^N}|w_\lambda|^q \\ &\ge \lambda^\sigma(\tfrac{1}{2}\|\nabla w_\lambda\|_2^2 - \tfrac{1}{q}\|w_\lambda\|_q^q) + \frac{m_0}{(\tau_1(w_\lambda))^{\frac{N}{\alpha}}} \\ &\ge \lambda^\sigma\frac{N(q-2)-4}{4q}\|w_\lambda\|_q^q + m_0 - C_1\lambda^\sigma\|w_\lambda\|_q^q. \end{aligned}$$

Recall that by Lemma 4.5, we have $m_0 - m_\lambda \ge C_2\lambda^\sigma$. Hence, we get

$$\frac{4-N(q-2)}{4q}\|w_\lambda\|_q^q \geq C_2 - C_1\|w_\lambda\|_q^q,$$

which yields

$$\|w_\lambda\|_q^q \geq \frac{4qC_2}{4-N(q-2)+4qC_1} > 0.$$

Since w_λ is bounded in $H^1(\mathbb{R}^N)$, it follows that $\|w_\lambda\|_q^q \sim 1$ as $\lambda \to 0$.

Since $\|w_\lambda\|_2^2 \sim 1$ as $\lambda \to 0$, the Gagliardo-Nirenberg inequality implies

$$\|w_\lambda\|_q^q \leq C\|\nabla w_\lambda\|_2^{\frac{N(q-2)}{2}},$$

which together with the boundedness of w_λ in $H^1(\mathbb{R}^N)$ yields $\|\nabla w_\lambda\|_2^2 \sim 1$ as $\lambda \to 0$.

Finally, by the definition of $\tau_1(w_\lambda)$, Lemma 4.3 and Lemma 4.4, it follows that

$$\mathbb{D}(w_\lambda) = \int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^{\frac{N+\alpha}{N}})|w_\lambda|^{\frac{N+\alpha}{N}} = (\tau_1(w_\lambda))^{-1}\|w_\lambda\|_2^2 \sim 1,$$

as $\lambda \to 0$. The proof is complete. □

Lemma 4.8. *Let $N \geq 3$ and $q \in (2, 2+\frac{4}{N})$, then for any $\lambda_n \to 0$, there exists $\rho \in [\rho_0, +\infty)$, such that, up to a subsequence, $w_{\lambda_n} \to U_\rho$ in $L^2(\mathbb{R}^N)$, where*

$$\rho_0 = \rho_0(q) := \left(\frac{2q\int_{\mathbb{R}^N}|\nabla U_1|^2}{N(q-2)\int_{\mathbb{R}^N}|U_1|^q}\right)^{\frac{2}{4-N(q-2)}}.$$

Moreover, $w_{\lambda_n} \to U_\rho$ in $D^{1,2}(\mathbb{R}^N)$ if and only if $\rho = \rho_0$.

Proof. Note that w_{λ_n} is a positive radially symmetric function, and by Lemma 4.2, $\{w_{\lambda_n}\}$ is bounded in $H^1(\mathbb{R}^N)$. Then there exists $w_0 \in H^1(\mathbb{R}^N)$ such that

$$w_{\lambda_n} \rightharpoonup w_0 \quad \text{weakly in } H^1(\mathbb{R}^N), \quad w_{\lambda_n} \to w_0 \quad \text{in } L^p(\mathbb{R}^N) \quad \text{for any } p \in (2, 2^*), \tag{4.14}$$

and

$$w_{\lambda_n}(x) \to w_0(x) \quad a.e. \text{ on } \mathbb{R}^N, \qquad w_{\lambda_n} \to w_0 \quad \text{in } L^2_{loc}(\mathbb{R}^N). \tag{4.15}$$

Observe that

$$J_0(w_{\lambda_n}) = J_{\lambda_n}(w_{\lambda_n}) + \frac{\lambda_n^\sigma}{q}\int_{\mathbb{R}^N}|w_{\lambda_n}|^q - \frac{\lambda_n^\sigma}{2}\int_{\mathbb{R}^N}|\nabla w_{\lambda_n}|^2 = m_{\lambda_n} + o_n(1) = m_0 + o_n(1),$$

and

$$J_0'(w_{\lambda_n})w = J_{\lambda_n}'(w_{\lambda_n})w + \lambda_n^\sigma \int_{\mathbb{R}^N} |w_{\lambda_n}|^{q-2} w_{\lambda_n} w - \lambda_n^\sigma \int_{\mathbb{R}^N} \nabla w_{\lambda_n} \nabla w = o_n(1).$$

Therefore, $\{w_{\lambda_n}\}$ is a PS sequence of J_0 at level $m_0 = \frac{\alpha}{2(N+\alpha)} S_1^{\frac{N+\alpha}{\alpha}}$.

By Lemma 4.7, we have $w_0 \neq 0$, and hence $m_0 \le J_0(w_0)$. By Lemma 4.5 and Lemma 4.6, we have

$$\begin{array}{rcl} o_n(1) & = & m_{\lambda_n} - m_0 \\ & \ge & \frac{\lambda_n^\sigma}{2} \|\nabla w_{\lambda_n}\|_2^2 + \frac{1}{2}[\|w_{\lambda_n}\|_2^2 - \|w_0\|_2^2] - \frac{N}{2(N+\alpha)}[\mathbb{D}(w_{\lambda_n}) - \mathbb{D}(w_0)] - \frac{\lambda_n^\sigma}{q}\|w_{\lambda_n}\|_q^q \\ & = & \frac{1}{2}\|w_{\lambda_n} - w_0\|_2^2 - \frac{N}{2(N+\alpha)}\mathbb{D}(w_{\lambda_n} - w_0) + o_n(1), \end{array}$$

$$\begin{array}{rcl} 0 & = & \langle J_{\lambda_n}'(w_{\lambda_n}) - J_0'(w_0), w_{\lambda_n} - w_0 \rangle \\ & = & \lambda_n^\sigma \int_{\mathbb{R}^N} \nabla w_{\lambda_n}(\nabla w_{\lambda_n} - \nabla w_0) + \int_{\mathbb{R}^N} |w_{\lambda_n} - w_0|^2 \\ & & - \int_{\mathbb{R}^N} (I_\alpha * |w_{\lambda_n}|^{\frac{N+\alpha}{N}}) w_{\lambda_n}^{\frac{\alpha}{N}} (w_{\lambda_n} - w_0) + \int_{\mathbb{R}^N} (I_\alpha * |w_0|^{\frac{N+\alpha}{N}}) w_0^{\frac{\alpha}{N}} (w_{\lambda_n} - w_0) \\ & & -\lambda_n^\sigma \int_{\mathbb{R}^N} |w_{\lambda_n}|^{q-2} w_{\lambda_n}(w_{\lambda_n} - w_0) \\ & = & \|w_{\lambda_n} - w_0\|_2^2 - \mathbb{D}(w_{\lambda_n} - w_0) + o_n(1). \end{array}$$

Hence, it follows that

$$\|w_{\lambda_n} - w_0\|_2^2 \le \frac{N}{N+\alpha}\mathbb{D}(w_{\lambda_n} - w_0) + o_n(1) = \frac{N}{N+\alpha}\|w_{\lambda_n} - w_0\|_2^2 + o_n(1),$$

and hence

$$\|w_{\lambda_n} - w_0\|_2 \to 0, \qquad \text{as } \lambda_n \to 0.$$

By the Hardy-Littlewood-Sobolev inequality and Lemma 4.6, it follows that

$$\lim_{\lambda_n \to 0} \mathbb{D}(w_{\lambda_n}) = \mathbb{D}(w_0).$$

Since $\tau_1(w_{\lambda_n}) \to 1$ as $\lambda_n \to 0$ by Lemma 4.3, it follows that $w_0 \in \mathcal{N}_0$.

On the other hand, by Lemma 4.1 and the boundedness of v_{λ_n} in $H^1(\mathbb{R}^N)$, we have

$$\begin{array}{rcl} m_{\lambda_n} & = & J_{\lambda_n}(w_{\lambda_n}) \\ & = & \frac{1}{2}\int_{\mathbb{R}^N} \lambda_n^\sigma |\nabla w_{\lambda_n}|^2 + |w_{\lambda_n}|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |w_{\lambda_n}|^{\frac{N+\alpha}{N}})|w_{\lambda_n}|^{\frac{N+\alpha}{N}} - \frac{\lambda_n^\sigma}{q}\int_{\mathbb{R}^N} |w_{\lambda_n}|^q \\ & = & \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 + |w_{\lambda_n}|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |w_{\lambda_n}|^{\frac{N+\alpha}{N}})|w_{\lambda_n}|^{\frac{N+\alpha}{N}} - \frac{\lambda_n}{q}\int_{\mathbb{R}^N} |v_{\lambda_n}|^q \\ & \ge & \frac{1}{2}\int_{\mathbb{R}^N} |w_{\lambda_n}|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |w_{\lambda_n}|^{\frac{N+\alpha}{N}})|w_{\lambda_n}|^{\frac{N+\alpha}{N}} - C\lambda_n. \end{array}$$

Sending $\lambda_n \to 0$, it then follows from Lemma 4.5 that

$$m_0 \ge \frac{1}{2}\int_{\mathbb{R}^N} |w_0|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |w_0|^{\frac{N+\alpha}{N}})|w_0|^{\frac{N+\alpha}{N}} = J_0(w_0).$$

Therefore, note that $w_0 \in \mathcal{N}_0$, we obtain $J_0(w_0) = m_0$. Thus, $w_0 = U_\rho$ for some $\rho \in (0, +\infty)$.

Moreover, by (4.5), we obtain

$$\|\nabla w_0\|_2^2 \le \lim_{\lambda_n \to 0} \|\nabla w_{\lambda_n}\|_2^2 = \frac{N(q-2)}{2q} \int_{\mathbb{R}^N} |w_0|^q,$$

from which it follows that

$$\rho \ge \left(\frac{2q \int_{\mathbb{R}^N} |\nabla U_1|^2}{N(q-2) \int_{\mathbb{R}^N} |U_1|^q} \right)^{\frac{2}{4-N(q-2)}}.$$

If $\rho = \rho_0$, then (4.5) implies that $\lim_{n\to\infty} \|\nabla w_{\lambda_n}\|_2^2 = \|\nabla U_{\rho_0}\|_2^2$, and hence $w_{\lambda_n} \to U_{\rho_0}$ in $D^{1,2}(\mathbb{R}^N)$. $\square$

Proof of Theorem 2.1. Let

$$M_\lambda = w_\lambda(0), \quad z_\lambda = M_\lambda [U_{\rho_0}(0)]^{-1},$$

where ρ_0 is given in Lemma 4.8. We further perform a scaling

$$\tilde{w}_\lambda(x) = z_\lambda^{-1} w_\lambda(z_\lambda^{-\frac{2}{N}} x),$$

then

$$\tilde{w}_\lambda(0) = z_\lambda^{-1} w_\lambda(0) = U_{\rho_0}(0) M_\lambda^{-1} w_\lambda(0) = U_{\rho_0}(0),$$

and $\tilde{w}_\lambda$ satisfies the rescaled equation

$$-\lambda^\sigma z_\lambda^{\frac{4}{N}} \Delta \tilde{w} + \tilde{w} = (I_\alpha * |\tilde{w}|^p)|\tilde{w}|^{p-2}\tilde{w} + \lambda^\sigma z_\lambda^{q-2} |\tilde{w}|^{q-2}\tilde{w}.$$

The corresponding functional is given by

$$J_\lambda(w_\lambda) = \frac{1}{2} \int_{\mathbb{R}^N} \lambda^\sigma z_\lambda^{\frac{4}{N}} |\nabla \tilde{w}_\lambda|^2 + |\tilde{w}_\lambda|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\tilde{w}_\lambda|^p)|\tilde{w}_\lambda|^p - \frac{1}{q} \lambda^\sigma z_\lambda^{q-2} \int_{\mathbb{R}^N} |\tilde{w}_\lambda|^q.$$

Moreover, we have

(1) $\|\tilde{w}\|_2^2 = \|w\|_2^2, \quad \int_{\mathbb{R}^N} (I_\alpha * |\tilde{w}|^p)|\tilde{w}|^p = \int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p,$

(2) $z_\lambda^{\frac{4}{N}} \|\nabla \tilde{w}\|_2^2 = \|\nabla w\|_2^2, \quad z_\lambda^{q-2} \|\tilde{w}\|_q^q = \|w\|_q^q.$

By Lemma 4.8, for any $\lambda_n \to 0$, there exists $\rho \ge \rho_0$ such that

$$M_{\lambda_n} = w_{\lambda_n}(0) \to U_\rho(0) = \rho^{-\frac{N}{2}} U_1(0) \le \rho_0^{-\frac{N}{2}} U_1(0) < +\infty,$$

which yields that $M_\lambda \le C$ for some $C > 0$ and any small $\lambda > 0$.

Suppose that there exists a sequence $\lambda_n \to 0$ such that $M_{\lambda_n} \to 0$. By Lemma 4.8, up to a subsequence, $M_{\lambda_n} = w_{\lambda_n}(0) \to U_\rho(0) \neq 0$ for some $\rho \in (0, +\infty)$. This leads to a contradiction. Therefore, there exists some $c > 0$ such that $M_\lambda \geq c > 0$.

Let

$$\zeta_\lambda = z_\lambda^{-\frac{2}{N}} \lambda^{-\frac{2}{4-N(q-2)}}.$$

Then

$$\zeta_\lambda \sim \lambda^{-\frac{2}{4-N(q-2)}}$$

and for small $\lambda > 0$, the rescaled family of ground states

$$\tilde{w}_\lambda(x) = \zeta_\lambda^{\frac{N}{2}} v_\lambda(\zeta_\lambda x)$$

satisfies

$$\|\nabla \tilde{w}_\lambda\|_2^2 \sim \|\tilde{w}_\lambda\|_q^q \sim \int_{\mathbb{R}^N} (I_\alpha * |\tilde{w}_\lambda|^{\frac{N+\alpha}{N}})|\tilde{w}_\lambda|^{\frac{N+\alpha}{N}} \sim \|\tilde{w}_\lambda\|_2^2 \sim 1,$$

and as $\lambda \to 0$, $\tilde{w}_\lambda$ converges in $L^2(\mathbb{R}^N)$ to the extremal function U_{ρ_0}. Then by Lemma 4.8, we also have $\tilde{w}_\lambda \to U_{\rho_0}$ in $D^{1,2}(\mathbb{R}^N)$. Thus we conclude that $\tilde{w}_\lambda \to U_{\rho_0}$ in $H^1(\mathbb{R}^N)$.

Since $w_\lambda \in \mathcal{N}_\lambda$, it follows that

$$\begin{aligned} m_\lambda &= (\tfrac{1}{2} - \tfrac{1}{2p})\lambda^\sigma \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + (\tfrac{1}{2} - \tfrac{1}{2p}) \int_{\mathbb{R}^N} |w_\lambda|^2 - (\tfrac{1}{q} - \tfrac{1}{2p})\lambda^\sigma \int_{\mathbb{R}^N} |w_\lambda|^q \\ &= \tfrac{\alpha}{2(N+\alpha)}\lambda^\sigma \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \tfrac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |w_\lambda|^2 - \tfrac{2p-q}{2pq}\lambda^\sigma \int_{\mathbb{R}^N} |w_\lambda|^q \\ &= \tfrac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |w_\lambda|^2 + O(\lambda^\sigma). \end{aligned}$$

Similarly, we also have

$$m_0 = \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |U_1|^2.$$

Then it follows from Lemma 4.5 that

$$\int_{\mathbb{R}^N} |U_1|^2 - \int_{\mathbb{R}^N} |w_\lambda|^2 = \frac{2(N+\alpha)}{\alpha}(m_0 - m_\lambda) + O(\lambda^\sigma) = O(\lambda^\sigma).$$

Since $\|U_1\|_2^2 = \int_{\mathbb{R}^N} (I_\alpha * |U_1|^p)|U_1|^p = S_1^{\frac{N+\alpha}{\alpha}}$, we conclude that

$$\|\tilde{w}_\lambda\|_2^2 = \|w_\lambda\|_2^2 = S_1^{\frac{N+\alpha}{\alpha}} + O(\lambda^\sigma).$$

Finally, by $w_\lambda \in \mathcal{N}_\lambda$, we also have

$$\int_{\mathbb{R}^N} (I_\alpha * |\tilde{w}_\lambda|^{\frac{N+\alpha}{N}})|\tilde{w}_\lambda|^{\frac{N+\alpha}{N}} = \int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^{\frac{N+\alpha}{N}})|w_\lambda|^{\frac{N+\alpha}{N}} = \|w_\lambda\|_2^2 + O(\lambda^\sigma) = S_1^{\frac{N+\alpha}{\alpha}} + O(\lambda^\sigma).$$

The statements on v_λ follow from the corresponding results on w_λ and $\tilde{w}_\lambda$, and the proof is complete. □

5. Proof of Theorem 2.2

In this section, we always assume that $p = \frac{N+\alpha}{N-2}$ and $q \in (2, 2^*)$ if $N \geq 4$, and $q \in (4, 6)$ if $N = 3$. It is easy to see that under the rescaling

$$w(x) = \lambda^{\frac{1}{q-2}} v(\lambda^{\frac{2^*-2}{2(q-2)}} x), \tag{5.1}$$

the equation (Q_λ) is reduced to

$$-\Delta w + \lambda^\sigma w = (I_\alpha * |w|^p)|w|^{p-2}w + \lambda^\sigma |w|^{q-2}w, \tag{$\bar{Q}_\lambda$}$$

where $\sigma := \frac{2^*-2}{q-2} > 1$. The associated functional is defined by

$$J_\lambda(w) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 + \lambda^\sigma |w|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p - \frac{1}{q}\lambda^\sigma \int_{\mathbb{R}^N} |w|^q.$$

Lemma 5.1. *Let $\lambda > 0$, $v \in H^1(\mathbb{R}^N)$ and w be the rescaling (5.1) of v. Then*
(1) $\|\nabla w\|_2^2 = \|\nabla v\|_2^2$, $\int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p = \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p$,
(2) $\lambda^\sigma \|w\|_2^2 = \|v\|_2^2$, $\lambda^\sigma \|w\|_q^q = \lambda \|v\|_q^q$,
(3) $I_\lambda(u) = J_\lambda(w)$.

We define the Nehari manifolds as follows:

$$\mathcal{N}_\lambda = \left\{ w \in H^1(\mathbb{R}^N) \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^N} |\nabla w|^2 + \lambda^\sigma \int_{\mathbb{R}^N} |w|^2 = \int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p + \lambda^\sigma \int_{\mathbb{R}^N} |w|^q \right\}$$

and

$$\mathcal{N}_0 = \left\{ w \in H^1(\mathbb{R}^N) \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^N} |\nabla w|^2 = \int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p \right\}.$$

Then

$$m_\lambda := \inf_{w \in \mathcal{N}_\lambda} J_\lambda(w), \quad \text{and} \quad m_0 := \inf_{u \in \mathcal{N}_0} J_0(u)$$

are well-defined and positive. Moreover, J_0 is attained on $\mathcal{N}_0$.

For $w \in H^1(\mathbb{R}^N) \setminus \{0\}$, we set

$$\tau_2(w) = \frac{\int_{\mathbb{R}^N} |\nabla w|^2}{\int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p}. \tag{5.2}$$

Then $(\tau_2(w))^{\frac{N-2}{2(2+\alpha)}} w \in \mathcal{N}_0$ for any $w \in H^1(\mathbb{R}^N) \setminus \{0\}$, and $w \in \mathcal{N}_0$ if and only if $\tau_2(w) = 1$.

Define the Pohožaev manifold as follows:

$$\mathcal{P}_\lambda := \{w \in H^1(\mathbb{R}^N) \setminus \{0\} \mid P_\lambda(w) = 0\},$$

where

$$\begin{aligned} P_\lambda(w) : \quad = \quad & \tfrac{N-2}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + \tfrac{\lambda^\sigma N}{2} \int_{\mathbb{R}^N} |w|^2 \\ & - \tfrac{N+\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p - \tfrac{\lambda^\sigma N}{q} \int_{\mathbb{R}^N} |w|^q. \end{aligned} \tag{5.3}$$

Then by Lemma 3.1, $w_\lambda \in \mathcal{P}_\lambda$. Moreover, we have a similar minimax characterizations for the least energy level m_λ as in Lemma 3.2.

Lemma 5.2. *The rescaled family of solutions $\{w_\lambda\}$ is bounded in $H^1(\mathbb{R}^N)$.*

Proof. First, we show that $\{w_\lambda\}$ is bounded in $H^1(\mathbb{R}^N)$. Since $\{w_\lambda\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$, it suffices to show that it is also bounded in $L^2(\mathbb{R}^N)$. By $w_\lambda \in \mathcal{N}_\lambda \cap \mathcal{P}_\lambda$, we obtain

$$\int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \lambda^\sigma \int_{\mathbb{R}^N} |w_\lambda|^2 = \int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^p)|w_\lambda|^p + \lambda^\sigma \int_{\mathbb{R}^N} |w_\lambda|^q,$$

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 + \frac{\lambda^\sigma N}{2} \int_{\mathbb{R}^N} |w_\lambda|^2 = \frac{N+\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^p)|w_\lambda|^p + \frac{\lambda^\sigma N}{q} \int_{\mathbb{R}^N} |w_\lambda|^q.$$

Thus, we obtain

$$\int_{\mathbb{R}^N} |w_\lambda|^2 = \frac{2(2^* - q)}{q(2^* - 2)} \int_{\mathbb{R}^N} |w_\lambda|^q. \tag{5.4}$$

By the Sobolev embedding theorem and the interpolation inequality, we obtain

$$\int_{\mathbb{R}^N} |w_\lambda|^q \le \left(\int_{\mathbb{R}^N} |w_\lambda|^2 \right)^{\frac{2^*-q}{2^*-2}} \left(\int_{\mathbb{R}^N} |w_\lambda|^{2^*} \right)^{\frac{q-2}{2^*-2}} \le \left(\int_{\mathbb{R}^N} |w_\lambda|^2 \right)^{\frac{2^*-q}{2^*-2}} \left(\frac{1}{S} \int_{\mathbb{R}^N} |\nabla w_\lambda|^2 \right)^{\frac{2^*(q-2)}{2(2^*-2)}},$$

where S is the best Sobolev constant. Therefore, we have

$$\left(\int_{\mathbb{R}^N} |w_\lambda|^2\right)^{\frac{q-2}{2^*-2}} \le \frac{2(2^*-q)}{q(2^*-2)}\left(\frac{1}{S}\int_{\mathbb{R}^N} |\nabla w_\lambda|^2\right)^{\frac{2^*(q-2)}{2(2^*-2)}}.$$

It then follows from Lemma 5.1 that

$$\int_{\mathbb{R}^N} |w_\lambda|^2 \le \left(\frac{2(2^*-q)}{q(2^*-2)}\right)^{\frac{2^*-2}{q-2}}\left(\frac{1}{S}\int_{\mathbb{R}^N} |\nabla v_\lambda|^2\right)^{2^*/2}, \tag{5.5}$$

which together with the boundedness of v_λ in $H^1(\mathbb{R}^N)$ implies that w_λ is bounded in $L^2(\mathbb{R}^N)$. □

Now, we give the following estimate on the least energy:

Lemma 5.3. *If $N \ge 5$ and $q \in (2, 2^*)$, then*

$$m_0 - m_\lambda \sim \lambda^\sigma \qquad as \quad \lambda \to 0.$$

If $N = 4$ and $q \in (2, 4)$, or $N = 3$ and $q \in (4, 6)$, then

$$m_0 - m_\lambda \lesssim \lambda^\sigma, \qquad as \quad \lambda \to 0.$$

Proof. First, we claim that there exists a constant $C > 0$ such that

$$1 < \tau_2(w_\lambda) \le 1 + C\lambda^\sigma. \tag{5.6}$$

In fact, since $w_\lambda \in \mathcal{N}_\lambda$, we see that

$$\tau_2(w_\lambda) = \frac{\int_{\mathbb{R}^N} |\nabla w_\lambda|^2}{\int_{\mathbb{R}^N}(I_\alpha * |w_\lambda|^p)|w_\lambda|^p} = 1 + \lambda^\sigma \frac{\int_{\mathbb{R}^N} |w_\lambda|^q - \int_{\mathbb{R}^N} |w_\lambda|^2}{\int_{\mathbb{R}^N}(I_\alpha * |w_\lambda|^p)|w_\lambda|^p}.$$

Since

$$\int_{\mathbb{R}^N} |w_\lambda|^q \le \left(\int_{\mathbb{R}^N} |w_\lambda|^2\right)^{\frac{2^*-q}{2^*-2}}\left(\int_{\mathbb{R}^N} |w_\lambda|^{2^*}\right)^{\frac{q-2}{2^*-2}},$$

we see that

$$\frac{\int_{\mathbb{R}^N} |w_\lambda|^q - \int_{\mathbb{R}^N} |w_\lambda|^2}{\int_{\mathbb{R}^N} |w_\lambda|^{2^*}} \le \zeta_\lambda^{\theta_q}(1-\zeta_\lambda^{1-\theta_q}) \le \theta_q^{\frac{\theta_q}{1-\theta_q}}(1-\theta_q) := G(q),$$

where

$$\theta_q = \frac{2^* - q}{2^* - 2}, \qquad \zeta_\lambda = \frac{\int_{\mathbb{R}^N} |v_\lambda|^2}{\int_{\mathbb{R}^N} |v_\lambda|^{2^*}}.$$

Therefore, by the boundedness of w_λ in $D^{1,2}(\mathbb{R}^N)$, we get

$$\begin{aligned} \tau_2(w_\lambda) &\le 1 + \lambda^\sigma G(q) \frac{\int_{\mathbb{R}^N} |w_\lambda|^{2^*}}{\int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^p)|w_\lambda|^p} \\ &\le 1 + \lambda^\sigma G(q) S^{-\frac{N}{N-2}} \frac{(\int_{\mathbb{R}^N} |\nabla w_\lambda|^2)^{\frac{N}{N-2}}}{\int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^p)|w_\lambda|^p} \\ &= 1 + \lambda^\sigma G(q) S^{-\frac{N}{N-2}} \tau_2(w_\lambda) (\int_{\mathbb{R}^N} |\nabla w_\lambda|^2)^{\frac{2}{N-2}} \\ &\le 1 + \lambda^\sigma C \tau_2(w_\lambda), \end{aligned}$$

and hence for small $\lambda > 0$, there holds

$$\tau_2(w_\lambda) \le \frac{1}{1 - \lambda^\sigma C} = 1 + \lambda^\sigma \frac{C}{1 - \lambda^\sigma C} \le 1 + \frac{1}{2} C \lambda^\sigma.$$

On the other hand, by (5.4), we have that $\int_{\mathbb{R}^N} |w_\lambda|^2 = \frac{2(2^*-q)}{q(2^*-2)} \int_{\mathbb{R}^N} |w_\lambda|^q < \int_{\mathbb{R}^N} |w_\lambda|^q$, therefore, we get $\tau_2(w_\lambda) > 1$. This proved the claim.

If $N \ge 3$, by Lemma 3.2 and the boundedness of $\{w_\lambda\}$, we find

$$\begin{aligned} m_0 &\le \sup_{t \ge 0} J_\lambda((w_\lambda)_t) + \lambda^\sigma t_\lambda^N \left(\frac{1}{q} \int_{\mathbb{R}^N} |w_\lambda|^q - \frac{1}{2} \int_{\mathbb{R}^N} |w_\lambda|^2 \right) \\ &\le m_\lambda + C\lambda^\sigma, \end{aligned} \tag{5.7}$$

where

$$t_\lambda = \left(\frac{\int_{\mathbb{R}^N} |\nabla w_\lambda|^2}{\int_{\mathbb{R}^N} (I_\alpha * |w_\lambda|^p)|w_\lambda|^p} \right)^{\frac{1}{2+\alpha}} = (\tau_2(w_\lambda))^{\frac{1}{2+\alpha}}.$$

For each $\rho > 0$, the family $V_\rho(x) := \rho^{-\frac{N-2}{2}} V_1(x/\rho)$ are radial ground states of $-\Delta v = (I_\alpha * |v|^p) v^{p-1}$, and verify that

$$\|V_\rho\|_2^2 = \rho^2 \|V_1\|_2^2, \qquad \int_{\mathbb{R}^N} |V_\rho|^q = \rho^{N - \frac{N-2}{2} q} \int_{\mathbb{R}^N} |V_1|^q. \tag{5.8}$$

Let $g_0(\rho) = \frac{1}{q} \int_{\mathbb{R}^N} |V_\rho|^q - \frac{1}{2} \int_{\mathbb{R}^N} |V_\rho|^2$. Then there exists $\rho_0 = \rho(q) \in (0, +\infty)$ with

$$\rho_0 = \left(\frac{[2N - q(N-2)] \int_{\mathbb{R}^N} |V_1|^q}{2q \int_{\mathbb{R}^N} |V_1|^2} \right)^{\frac{2}{(N-2)(q-2)}}$$

such that

$$g_0(\rho_0) = \sup_{\rho > 0} g_0(\rho) = \frac{(N-2)(q-2)}{4q} \left(\frac{[2N - q(N-2)] \int_{\mathbb{R}^N} |V_1|^q}{2q \int_{\mathbb{R}^N} |V_1|^2} \right)^{\frac{2^*-q}{q-2}} \int_{\mathbb{R}^N} |V_1|^q.$$

Let $V_0 = V_{\rho_0}$, then there exists $t_\lambda \in (0, +\infty)$ such that

$$\begin{aligned} m_\lambda &\le \sup_{t\ge 0} J_\lambda(tV_0) = J_\lambda(t_\lambda V_0) \\ &= \frac{t_\lambda^2}{2}\int_{\mathbb{R}^N} |\nabla V_0|^2 - \frac{t_\lambda^{2p}}{2p}(I_\alpha * |V_0|^p)|V_0|^p - \lambda^\sigma\{\frac{t_\lambda^q}{q}\int_{\mathbb{R}^N}|V_0|^q - \frac{t_\lambda^2}{2}\int_{\mathbb{R}^N}|V_0|^2\} \\ &\le \sup_{t\ge 0}\left(\frac{t^2}{2} - \frac{t^{2p}}{2p}\right)\int_{\mathbb{R}^N}|\nabla V_0|^2 - \lambda^\sigma\{\frac{t_\lambda^q}{q}\int_{\mathbb{R}^N}|V_0|^q - \frac{t_\lambda^2}{2}\int_{\mathbb{R}^N}|V_0|^2\} \\ &= m_0 - \lambda^\sigma\{\frac{t_\lambda^q}{q}\int_{\mathbb{R}^N}|V_0|^q - \frac{t_\lambda^2}{2}\int_{\mathbb{R}^N}|V_0|^2\}. \end{aligned} \tag{5.9}$$

If $t_\lambda \ge 1$, then

$$\int_{\mathbb{R}^N} |\nabla V_0|^2 + \lambda^\sigma \int_{\mathbb{R}^N} |V_0|^2 \ge t_\lambda^{q-2}\left\{\int_{\mathbb{R}^N} (I_\alpha * |V_0|^p)|V_0|^p + \lambda^\sigma \int_{\mathbb{R}^N}|V_0|^q\right\}.$$

Hence

$$t_\lambda \le \max\left\{1, \left(\frac{\int_{\mathbb{R}^N}|\nabla V_0|^2 + \lambda^\sigma\int_{\mathbb{R}^N}|V_0|^2}{\int_{\mathbb{R}^N}(I_\alpha * |V_0|^p)|V_0|^p + \lambda^\sigma\int_{\mathbb{R}^N}|V_0|^q}\right)^{\frac{1}{q-2}}\right\}.$$

If $t_\lambda \le 1$, then

$$\int_{\mathbb{R}^N} |\nabla V_0|^2 + \lambda^\sigma \int_{\mathbb{R}^N} |V_0|^2 \le t_\lambda^{q-2}\left\{\int_{\mathbb{R}^N} (I_\alpha * |V_0|^p)|V_0|^p + \lambda^\sigma \int_{\mathbb{R}^N}|V_0|^q\right\}.$$

Hence

$$t_\lambda \ge \min\left\{1, \left(\frac{\int_{\mathbb{R}^N}|\nabla V_0|^2 + \lambda^\sigma\int_{\mathbb{R}^N}|V_0|^2}{\int_{\mathbb{R}^N}(I_\alpha * |V_0|^p)|V_0|^p + \lambda^\sigma\int_{\mathbb{R}^N}|V_0|^q}\right)^{\frac{1}{q-2}}\right\}.$$

Since

$$\int_{\mathbb{R}^N}(I_\alpha * |V_0|^p)|V_0|^p = \int_{\mathbb{R}^N}|\nabla V_0|^2 \qquad \text{and} \qquad \int_{\mathbb{R}^N}|V_0|^q > \int_{\mathbb{R}^N}|V_0|^2,$$

we conclude that

$$\left(\frac{\int_{\mathbb{R}^N}|\nabla V_0|^2 + \lambda^\sigma\int_{\mathbb{R}^N}|V_0|^2}{\int_{\mathbb{R}^N}(I_\alpha * |V_0|^p)|V_0|^p + \lambda^\sigma\int_{\mathbb{R}^N}|V_0|^q}\right)^{\frac{1}{q-2}} \le t_\lambda \le 1. \tag{5.10}$$

Therefore, $\lim_{\lambda\to 0} t_\lambda = 1$ and hence there exists a constant $C > 0$ such that

$$m_\lambda \le m_0 - C\lambda^\sigma,$$

for small $\lambda > 0$. The proof is complete. $\square$

Lemma 5.4. *In the lower dimension cases, there exists a constant $\varpi = \varpi(q) > 0$ such that for $\lambda > 0$ small,*

$$m_\lambda \leq \begin{cases} m_0 - \lambda^\sigma \left(\ln \frac{1}{\lambda}\right)^{-\frac{4-q}{q-2}} \varpi = m_0 - \lambda^{\frac{2}{q-2}} \left(\ln \frac{1}{\lambda}\right)^{-\frac{4-q}{q-2}} \varpi, & \text{if} \quad N = 4, \\ m_0 - \lambda^{\sigma + \frac{2(6-q)}{(q-4)(q-2)}} \varpi = m_0 - \lambda^{\frac{2}{q-4}} \varpi, & \text{if} \quad N = 3 \text{ and } q \in (4, 6). \end{cases}$$

Proof. Let $\rho > 0$ and R be a large parameter, and $\eta_R \in C_0^\infty(\mathbb{R})$ is a cut-off function such that $\eta_R(r) = 1$ for $|r| < R$, $0 < \eta_R(r) < 1$ for $R < |r| < 2R$, $\eta_R(r) = 0$ for $|r| > 2R$ and $|\eta_R'(r)| \leq 2/R$.

For $\ell \gg 1$, a straightforward computation shows that

$$\int_{\mathbb{R}^N} |\nabla(\eta_\ell V_1)|^2 = \begin{cases} \frac{2(N+\alpha)}{2+\alpha} m_0 + O(\ell^{-2}), & \text{if} \quad N = 4, \\ \frac{2(N+\alpha)}{2+\alpha} m_0 + O(\ell^{-1}), & \text{if} \quad N = 3. \end{cases}$$

$$\int_{\mathbb{R}^N} (I_\alpha * |\eta_\ell V_1|^p)|\eta_\ell V_1|^p = \frac{2(N+\alpha)}{2+\alpha} m_0 + O(\ell^{-N}),$$

$$\int_{\mathbb{R}^N} |\eta_\ell V_1|^2 = \begin{cases} \ln \ell (1 + o(1)), & \text{if} \quad N = 4, \\ \ell(1 + o(1)), & \text{if} \quad N = 3. \end{cases}$$

By Lemma 3.2, we find

$$\begin{aligned} m_\lambda &\leq \sup_{t \geq 0} J_\lambda((\eta_R V_\rho)_t) = J_\lambda((\eta_R V_\rho)_{t_\lambda}) \\ &\leq \sup_{t \geq 0} \left(\frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla(\eta_R V_\rho)|^2 - \frac{t^{N+\alpha}}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\eta_R V_\rho|^p)|\eta_R V_\rho|^p \right) \\ &\quad - \lambda^\sigma t_\lambda^N \left[\int_{\mathbb{R}^N} \frac{1}{q} |\eta_R V_\rho|^q - \frac{1}{2} |\eta_R V_\rho|^2 \right] \\ &= (I) - \lambda^\sigma (II), \end{aligned} \tag{5.11}$$

where $t_\lambda \in (0, +\infty)$ is the unique critical point of the function $g(t)$ defined by

$$\begin{aligned} g(t) = &\frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla(\eta_R V_\rho)|^2 + \frac{t^N}{2} \lambda^\sigma \int_{\mathbb{R}^N} |\eta_R V_\rho|^2 \\ &- \frac{t^{N+\alpha}}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\eta_R V_\rho|^p)|\eta_R V_\rho|^p - \frac{t^N}{q} \lambda^\sigma \int_{\mathbb{R}^N} |\eta_R V_\rho|^q. \end{aligned}$$

That is, $t = t_\lambda$ solves the equation $\ell_1(t) = \ell_2(t)$, where

$$\ell_1(t) := \frac{N-2}{2t^2} \int_{\mathbb{R}^N} |\nabla(\eta_R V_\rho)|^2$$

and

$$\ell_2(t) := \frac{N+\alpha}{2p} t^\alpha \int_{\mathbb{R}^N} (I_\alpha * |\eta_R V_\rho|^p)|\eta_R V_\rho|^p + \frac{N}{q} \lambda^\sigma \int_{\mathbb{R}^N} |\eta_R V_\rho|^q - \frac{N}{2} \lambda^\sigma \int_{\mathbb{R}^N} |\eta_R V_\rho|^2.$$

If $t_\lambda \geq 1$, then

$$\begin{aligned}\frac{N-2}{2t_\lambda^2}\int_{\mathbb{R}^N}|\nabla(\eta_R V_\rho)|^2 \quad &\geq \quad \frac{N+\alpha}{2p}\int_{\mathbb{R}^N}(I_\alpha*|\eta_R V_\rho|^p)|\eta_R V_\rho|^p \\ &\quad +\frac{N}{q}\lambda^\sigma\int_{\mathbb{R}^N}|\eta_R V_\rho|^q-\frac{N}{2}\lambda^\sigma\int_{\mathbb{R}^N}|\eta_R V_\rho|^2,\end{aligned}$$

and hence

$$\begin{aligned} t_\lambda \quad &\leq \quad \left(\frac{\int_{\mathbb{R}^N}|\nabla(\eta_R V_\rho)|^2}{\int_{\mathbb{R}^N}(I_\alpha*|\eta_R V_\rho|^p)|\eta_R V_\rho|^p+2^*\lambda^\sigma\{\frac{1}{q}\int_{\mathbb{R}^N}|\eta_R V_\rho|^q-\frac{1}{2}\int_{\mathbb{R}^N}|\eta_R V_\rho|^2\}}\right)^{\frac{1}{2}} \\ &\leq \quad \left(\frac{\int_{\mathbb{R}^N}|\nabla(\eta_R V_\rho)|^2}{\int_{\mathbb{R}^N}(I_\alpha*|\eta_R V_\rho|^p)|\eta_R V_\rho|^p}\right)^{\frac{1}{2}}.\end{aligned} \tag{5.12}$$

If $t_\lambda \leq 1$, then

$$\begin{aligned}\frac{N-2}{2t_\lambda^2}\int_{\mathbb{R}^N}|\nabla(\eta_R V_\rho)|^2 \quad &\leq \quad \frac{N+\alpha}{2p}\int_{\mathbb{R}^N}(I_\alpha*|\eta_R V_\rho|^p)|\eta_R V_\rho|^p \\ &\quad +\frac{N}{q}\lambda^\sigma\int_{\mathbb{R}^N}|\eta_R V_\rho|^q-\frac{N}{2}\lambda^\sigma\int_{\mathbb{R}^N}|\eta_R V_\rho|^2,\end{aligned}$$

and hence

$$\begin{aligned} t_\lambda \quad &\geq \quad \left(\frac{\int_{\mathbb{R}^N}|\nabla(\eta_R V_\rho)|^2}{\int_{\mathbb{R}^N}(I_\alpha*|\eta_R V_\rho|^p)|\eta_R V_\rho|^p+2^*\lambda^\sigma\{\frac{1}{q}\int_{\mathbb{R}^N}|\eta_R V_\rho|^q-\frac{1}{2}\int_{\mathbb{R}^N}|\eta_R V_\rho|^2\}}\right)^{\frac{1}{2}} \\ &\geq \quad \left(\frac{\int_{\mathbb{R}^N}|\nabla(\eta_R V_\rho)|^2}{\int_{\mathbb{R}^N}(I_\alpha*|\eta_R V_\rho|^p)|\eta_R V_\rho|^p}\right)^{\frac{1}{2}}\left\{1-2^*\lambda^\sigma\frac{\frac{1}{q}\int_{\mathbb{R}^N}|\eta_R V_\rho|^q-\frac{1}{2}\int_{\mathbb{R}^N}|\eta_R V_\rho|^2}{\int_{\mathbb{R}^N}(I_\alpha*|\eta_R V_\rho|^p)|\eta_R V_\rho|^p}\right\}.\end{aligned} \tag{5.13}$$

Therefore, we obtain

$$\begin{aligned}|t_\lambda-1| \quad &\leq \quad \left|\left(\frac{\int_{\mathbb{R}^N}|\nabla(\eta_R V_\rho)|^2}{\int_{\mathbb{R}^N}(I_\alpha*|\eta_R V_\rho|^p)|\eta_R V_\rho|^p}\right)^{\frac{1}{2}}-1\right| \\ &\quad +\lambda^\sigma\left(\frac{\int_{\mathbb{R}^N}|\nabla(\eta_R V_\rho)|^2}{\int_{\mathbb{R}^N}(I_\alpha*|\eta_R V_\rho|^p)|\eta_R V_\rho|^p}\right)^{\frac{1}{2}}\frac{2^*\phi(\rho)}{\int_{\mathbb{R}^N}(I_\alpha*|\eta_R V_\rho|^p)|\eta_R V_\rho|^p},\end{aligned}$$

where $\phi(\rho):=\frac{1}{q}\int_{\mathbb{R}^N}|\eta_R V_\rho|^q-\frac{1}{2}\int_{\mathbb{R}^N}|\eta_R V_\rho|^2$.

Set $\ell = R/\rho$, then

$$\begin{aligned}(I) \quad &= \quad \frac{2+\alpha}{2(N+\alpha)}\frac{(\int_{\mathbb{R}^N}|\nabla(\eta_\ell V_1)|^2)^{\frac{N+\alpha}{2+\alpha}}}{(\int_{\mathbb{R}^N}(I_\alpha*|\eta_\ell V_1|^p)|\eta_\ell V_1|^p)^{\frac{N-2}{2+\alpha}}} \\ &= \quad \begin{cases}\frac{2+\alpha}{2(N+\alpha)}\frac{(\frac{2(N+\alpha)}{2+\alpha}m_0+O(\ell^{-2}))^{\frac{N+\alpha}{2+\alpha}}}{(\frac{2(N+\alpha)}{2+\alpha}m_0+O(\ell^{-4}))^{\frac{N-2}{2+\alpha}}}, & \text{if } N=4,\\ \frac{2+\alpha}{2(N+\alpha)}\frac{(\frac{2(N+\alpha)}{2+\alpha}m_0+O(\ell^{-1}))^{\frac{N+\alpha}{2+\alpha}}}{(\frac{2(N+\alpha)}{2+\alpha}m_0+O(\ell^{-3}))^{\frac{N-2}{2+\alpha}}}, & \text{if } N=3,\end{cases} \\ &= \quad \begin{cases} m_0+O(\ell^{-2}), & \text{if } N=4,\\ m_0+O(\ell^{-1}), & \text{if } N=3.\end{cases}\end{aligned} \tag{5.14}$$

Since

$$\begin{aligned}\phi(\rho) &= \tfrac{1}{q}\int_{\mathbb{R}^N}|\eta_R V_\rho|^q - \tfrac{1}{2}\int_{\mathbb{R}^N}|\eta_R V_\rho|^2\\ &= \tfrac{1}{q}\rho^{N-\frac{N-2}{2}q}\int_{\mathbb{R}^N}|\eta_\ell V_1|^q - \tfrac{1}{2}\rho^2\int_{\mathbb{R}^N}|\eta_\ell V_1|^2\end{aligned}$$

take its maximum value $\varphi(\rho_\ell)$ at the unique point

$$\begin{aligned}\rho_\ell : &= \left(\frac{[2N-q(N-2)]\int_{\mathbb{R}^N}|\eta_\ell V_1|^q}{2q\int_{\mathbb{R}^N}|\eta_\ell V_1|^2}\right)^{\frac{2}{(N-2)(q-2)}}\\ &\sim \begin{cases}(\ln\ell)^{-\frac{2}{(N-2)(q-2)}} & \text{if} \quad N=4,\\ \ell^{-\frac{2}{(N-2)(q-2)}} & \text{if} \quad N=3,\end{cases}\end{aligned}$$

we obtain

$$\begin{aligned}\phi(\rho_\ell) &= \sup_{\rho\ge 0}\phi(\rho)\\ &= \frac{4+q(N-2)-2N}{4q}\rho_\ell^{N-\frac{N-2}{2}q}\int_{\mathbb{R}^N}|\eta_\ell V_1|^q\\ &= \frac{4+q(N-2)-2N}{4q}\left(\frac{2N-q(N-2)}{2q}\right)^{\frac{2N-q(N-2)}{(N-2)(q-2)}}\frac{\left(\int_{\mathbb{R}^N}|\eta_\ell V_1|^q\right)^{\frac{4}{(N-2)(q-2)}}}{\left(\int_{\mathbb{R}^N}|\eta_\ell V_1|^2\right)^{\frac{2N-q(N-2)}{(N-2)(q-2)}}}\\ &\le \frac{4+q(N-2)-2N}{4q}\left(\frac{2N-q(N-2)}{2q}\right)^{\frac{2N-q(N-2)}{(N-2)(q-2)}}\int_{\mathbb{R}^N}|\eta_\ell V_1|^{2^*}\\ &\to \frac{4+q(N-2)-2N}{4q}\left(\frac{2N-q(N-2)}{2q}\right)^{\frac{2N-q(N-2)}{(N-2)(q-2)}}\int_{\mathbb{R}^N}|V_1|^{2^*},\end{aligned}$$

as $\ell\to+\infty$, where we have used the interpolation inequality

$$\int_{\mathbb{R}^N}|\eta_\ell V_1|^q \le \left(\int_{\mathbb{R}^N}|\eta_\ell V_1|^2\right)^{\frac{2^*-q}{2^*-2}}\left(\int_{\mathbb{R}^N}|\eta_\ell V_1|^{2^*}\right)^{\frac{q-2}{2^*-2}}.$$

Since

$$\int_{\mathbb{R}^N}|\eta_\ell V_1|^q \to \int_{\mathbb{R}^N}|V_1|^q,$$

as $\ell\to+\infty$, it follows that

$$\begin{aligned}\phi(\rho_\ell) &= \frac{4+q(N-2)-2N}{4q}\left(\frac{[2N-q(N-2)]\int_{\mathbb{R}^N}|\eta_\ell V_1|^q}{2q\int_{\mathbb{R}^N}|\eta_\ell V_1|^2}\right)^{\frac{2N-q(N-2)}{(N-2)(q-2)}}\int_{\mathbb{R}^N}|\eta_\ell V_1|^q\\ &= \begin{cases}C(\ln\ell(1+o(1))^{-\frac{2N-q(N-2)}{(N-2)(q-2)}} & \text{if} \quad N=4,\\ C(\ell(1+o(1))^{-\frac{2N-q(N-2)}{(N-2)(q-2)}} & \text{if} \quad N=3.\end{cases}\end{aligned}$$

Since $\phi(\rho)$ is bounded, we find

$$
\begin{aligned}
|t_\lambda - 1| &\le \left| \left(\frac{\int_{\mathbb{R}^N} |\nabla(\eta_\ell V_1)|^2}{\int_{\mathbb{R}^N} (I_\alpha * |\eta_\ell V_1|^p)|\eta_\ell V_1|^p} \right)^{\frac{1}{2}} - 1 \right| \\
&\quad + \lambda^\sigma \left(\frac{\int_{\mathbb{R}^N} |\nabla(\eta_\ell V_1)|^2}{\int_{\mathbb{R}^N} (I_\alpha * |\eta_\ell V_1|^p)|\eta_\ell V_1|^p} \right)^{\frac{1}{2}} \frac{2^* C}{\int_{\mathbb{R}^N} (I_\alpha * |\eta_\ell V_1|^p)|\eta_\ell V_1|^p} \\
&\to \frac{2^* C \lambda^\sigma}{\int_{\mathbb{R}^N} (I_\alpha * |V_1|^p)|V_1|^p},
\end{aligned}
$$

as $\ell \to +\infty$. Thus, for small $\lambda > 0$, we have

$$
\begin{aligned}
(II) &= \phi(\rho_\ell) + (t_\lambda^N - 1)\phi(\rho_\ell) \\
&\sim \begin{cases} (\ln \ell)^{-\frac{2N - q(N-2)}{(N-2)(q-2)}}, & \text{if} \quad N = 4, \\ \ell^{-\frac{2N - q(N-2)}{(N-2)(q-2)}}, & \text{if} \quad N = 3. \end{cases}
\end{aligned}
$$

It follows that if $N = 4$, then

$$
\begin{aligned}
m_\lambda &\le (I) - \lambda^\sigma (II) \\
&\le m_0 + O(\ell^{-2}) - C\lambda^\sigma (\ln \ell)^{-\frac{2N - q(N-2)}{(N-2)(q-2)}}.
\end{aligned} \tag{5.15}
$$

Take $\ell = (1/\lambda)^M$. Then

$$
m_\lambda \le m_0 + O(\lambda^{2M}) - C\lambda^\sigma M^{-\frac{2N - q(N-2)}{(N-2)(q-2)}} (\ln \frac{1}{\lambda})^{-\frac{2N - q(N-2)}{(N-2)(q-2)}}.
$$

If $M > \frac{1}{q-2}$, then $2M > \sigma$, and hence

$$
m_\lambda \le m_0 - \lambda^\sigma (\ln \frac{1}{\lambda})^{-\frac{2N - q(N-2)}{(N-2)(q-2)}} \varpi = m_0 - \lambda^{\frac{2}{q-2}} (\ln \frac{1}{\lambda})^{-\frac{4-q}{q-2}} \varpi, \tag{5.16}
$$

for small $\lambda > 0$, where

$$
\varpi = \frac{1}{2} C M^{-\frac{2N - q(N-2)}{(N-2)(q-2)}}.
$$

If $N = 3$, then

$$
\begin{aligned}
m_\lambda &\le (I) - \lambda^\sigma (II) \\
&\le m_0 + O(\ell^{-1}) - C\lambda^\sigma \ell^{-\frac{2N - q(N-2)}{(N-2)(q-2)}}.
\end{aligned} \tag{5.17}
$$

Take $\ell = \delta^{-1}\lambda^{-\tau}$. Then

$$
m_\lambda \le m_0 + \delta O(\lambda^\tau) - C\lambda^\sigma \delta^{\frac{2N - q(N-2)}{(N-2)(q-2)}} \lambda^{\tau \frac{2N - q(N-2)}{(N-2)(q-2)}}
$$

If $q \in (4, 6)$ and

$$
\tau = \frac{2(N-2)}{2 + q(N-2) - 2N} = \frac{2}{q-4},
$$

then

$$m_\lambda \le m_0 + (\delta O(1) - C\delta^{\frac{2N-q(N-2)}{(N-2)(q-2)}})\lambda^{\frac{2}{q-4}}.$$

Since

$$1 > \frac{2N-q(N-2)}{(N-2)(q-2)},$$

it follows that for some small $\delta > 0$, there exists $\varpi > 0$ such that

$$m_\lambda \le m_0 - \lambda^{\frac{2}{q-4}}\varpi.$$

This completes the proof. □

Combining Lemma 5.3 and Lemma 5.4, we get the following.

Lemma 5.5. *Let* $\delta_\lambda := m_0 - m_\lambda$, *then*

$$\lambda^\sigma \gtrsim \delta_\lambda \gtrsim \begin{cases} \lambda^\sigma, & \text{if} \quad N \ge 5, \\ \lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}, & \text{if} \quad N = 4, \\ \lambda^{\frac{2}{q-4}}, & \text{if} \quad N = 3 \text{ and } q \in (4,6). \end{cases}$$

Lemma 5.6. *Assume* $N \ge 5$. *Then* $\|w_\lambda\|_q \sim 1$ *as* $\lambda \to 0$.

Proof. By (5.7), we have

$$m_0 \le m_\lambda + \lambda^\sigma(\tau_2(w_\lambda))^{\frac{N}{2+\alpha}}\frac{q-2}{q(2^*-2)}\int_{\mathbb{R}^N}|w_\lambda|^q.$$

Therefore, it follows from (5.6) and Lemma 5.5 that

$$\|w_\lambda\|_q^q \ge \frac{m_0 - m_\lambda}{(\tau_2(w_\lambda))^{\frac{N}{2+\alpha}}}\cdot\frac{q(2^*-2)}{q-2}\lambda^{-\sigma} \ge \frac{Cq(2^*-2)}{(q-2)(\tau_2(w_\lambda))^{\frac{N}{2+\alpha}}} \ge C > 0,$$

which together with the boundedness of $\{w_\lambda\}$ implies the desired conclusion. □

Lemma 5.7. *Let* $N \ge 5$, $\alpha > N-4$ *and* $q \in (2, 2^*)$, *then* $w_\lambda \to V_{\rho_0}$ *in* $H^1(\mathbb{R}^N)$ *as* $\lambda \to 0$, *where* V_{ρ_0} *is a positive ground state of the equation* $-\Delta V = (I_\alpha * |V|^p)V^{p-1}$ *with*

$$\rho_0 = \left(\frac{2(2^*-q)\int_{\mathbb{R}^N}|V_1|^q}{q(2^*-2)\int_{\mathbb{R}^N}|V_1|^2}\right)^{\frac{2}{(N-2)(q-2)}}. \tag{5.18}$$

In the lower dimension cases $N = 4$ *and* $N = 3$, *there exists* $\xi_\lambda \in (0, +\infty)$ *with* $\xi_\lambda \to 0$ *such that*

$$w_\lambda - \xi_\lambda^{-\frac{N-2}{2}}V_1(\xi_\lambda^{-1}\cdot) \to 0$$

as $\lambda \to 0$ *in* $D^{1,2}(\mathbb{R}^N)$ *and* $L^{2^*}(\mathbb{R}^N)$.

Proof. Note that w_λ is a positive radially symmetric function, and by Lemma 5.2, $\{w_\lambda\}$ is bounded in $H^1(\mathbb{R}^N)$. Then there exists $w_0 \in H^1(\mathbb{R}^N)$ verifying $-\Delta w = (I_\alpha * w^p)w^{p-1}$ such that

$$w_\lambda \rightharpoonup w_0 \quad \text{weakly in } H^1(\mathbb{R}^N), \quad w_\lambda \to w_0 \quad \text{in } L^p(\mathbb{R}^N) \quad \text{for any } p \in (2, 2^*), \tag{5.19}$$

and

$$w_\lambda(x) \to w_0(x) \quad a.\,e. \text{ on } R^N, \qquad w_\lambda \to w_0 \quad \text{in } L^2_{loc}(\mathbb{R}^N). \tag{5.20}$$

Observe that

$$J_0(w_\lambda) = J_\lambda(w_\lambda) + \frac{\lambda^\sigma}{q}\int_{\mathbb{R}^N} |w_\lambda|^q - \frac{\lambda^\sigma}{2}\int_{\mathbb{R}^N} |w_\lambda|^2 = m_\lambda + o(1) = m_0 + o(1),$$

and

$$J_0'(w_\lambda)w = J_\lambda'(w_\lambda)w + \lambda^\sigma \int_{\mathbb{R}^N} |w_\lambda|^{q-2} w_\lambda w - \lambda^\sigma \int_{\mathbb{R}^N} w_\lambda w = o(1).$$

Therefore, $\{w_\lambda\}$ is a PS sequence of J_0 at level $m_0 = \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}$.

By Lemma 3.6, it is standard [46] to show that there exists $\zeta_\lambda^{(j)} \in (0, +\infty)$, $w^{(j)} \in D^{1,2}(\mathbb{R}^N)$ with $j = 1, 2, \cdots, k$, k a non-negative integer, such that

$$w_\lambda = w_0 + \sum_{j=1}^{k} (\zeta_\lambda^{(j)})^{-\frac{N-2}{2}} w^{(j)}((\zeta_\lambda^{(j)})^{-1}x) + \tilde{w}_\lambda, \tag{5.21}$$

where $\tilde{w}_\lambda \to 0$ in $L^{2^*}(\mathbb{R}^N)$ and $w^{(j)}$ are nontrivial solutions of the limit equation $-\Delta v = (I_\alpha * v^p)v^{p-1}$. Moreover, we have

$$\limsup_{\lambda \to 0} \|w_\lambda\|^2_{D^1(\mathbb{R}^N)} \ge \|w_0\|^2_{D^1(\mathbb{R}^N)} + \sum_{j=1}^{k} \|w^{(j)}\|^2_{D^1(\mathbb{R}^N)} \tag{5.22}$$

and

$$m_0 = J_0(w_0) + \sum_{j=1}^{k} J_0(w^{(j)}). \tag{5.23}$$

Moreover, $J_0(w_0) \ge 0$ and $J_0(w^{(j)}) \ge m_0$ for all $j = 1, 2, \cdots, k$.

If $N \ge 5$, then by Lemma 5.6, we have $w_0 \ne 0$ and hence $J_0(w_0) = m_0$ and $k = 0$. Thus $w_\lambda \to w_0$ in $L^{2^*}(\mathbb{R}^N)$. Since $J_0'(w_\lambda) \to 0$, it follows that $w_\lambda \to w_0$ in $D^{1,2}(\mathbb{R}^N)$.

Since $w_\lambda(x)$ is radial and radially decreasing, for every $x \in \mathbb{R}^N \setminus \{0\}$, we have

$$w_\lambda^2(x) \le \frac{1}{|B_{|x|}|} \int\limits_{B_{|x|}} |w_\lambda|^2 \le \frac{1}{|x|^N} \int\limits_{\mathbb{R}^N} |w_\lambda|^2 \le \frac{C}{|x|^N},$$

then

$$w_\lambda(x) \le C|x|^{-\frac{N}{2}}, \qquad |x| \ge 1. \tag{5.24}$$

If $\alpha > N-4$, then we have $p = \frac{N+\alpha}{N-2} > 2$ and hence

$$|w_\lambda|^p |x|^N \le C|x|^{-\frac{N}{2}p+N} = C|x|^{-\frac{N}{2}(p-2)} \to 0, \qquad \text{as } |x| \to +\infty.$$

By virtue of Lemma 3.10, we obtain

$$(I_\alpha * |w_\lambda|^p)(x) \le C|x|^{-N+\alpha}, \qquad |x| \ge 1,$$

and then

$$(I_\alpha * |w_\lambda|^p)(x)|w_\lambda|^{p-2}(x) \le C|x|^{-\frac{N^2-N\alpha+4\alpha}{2(N-2)}}, \qquad |x| \ge \tilde{R}. \tag{5.25}$$

Since

$$\left(-\Delta - C|x|^{-\frac{N^2-N\alpha+4\alpha}{2(N-2)}}\right) w_\lambda \le \left(-\Delta + \lambda^\sigma - (I_\alpha * |w_\lambda|^p) w_\lambda^{p-2} - \lambda^\sigma w_\lambda^{q-2}\right) w_\lambda = 0,$$

for large $|x|$. We also have

$$\left(-\Delta - C|x|^{-\frac{N^2-N\alpha+4\alpha}{2(N-2)}}\right) \frac{1}{|x|^{N-2-\varepsilon_0}} = \left(\varepsilon_0(N-2-\varepsilon_0) - C|x|^{-\frac{(N-4)(N-\alpha)+8}{2(N-2)}}\right) \frac{1}{|x|^{N-\varepsilon_0}},$$

which is positive for $|x|$ large enough. By (5.24) and the maximum principle on $\mathbb{R}^N \setminus B_R$, we deduce that

$$w_\lambda(x) \le \frac{w_\lambda(R) R^{N-2-\varepsilon_0}}{|x|^{N-2-\varepsilon_0}} \le \frac{C R^{N/2-2-\varepsilon_0}}{|x|^{N-2-\varepsilon_0}}, \qquad \text{for } |x| \ge R. \tag{5.26}$$

When $\varepsilon_0 > 0$ is small enough, the domination is in $L^2(\mathbb{R}^N)$ for $N \ge 5$, and this shows, by the dominated convergence theorem, that $w_\lambda \to w_0$ in $L^2(\mathbb{R}^N)$. Thus, we conclude that $w_\lambda \to w_0$ in $H^1(\mathbb{R}^N)$. Moreover, by (5.4), we obtain

$$\|w_0\|_2^2 = \frac{2(2^*-q)}{q(2^*-2)} \int\limits_{\mathbb{R}^N} |w_0|^q,$$

from which it follows that $w_0 = V_{\rho_0}$ with

$$\rho_0=\left(\frac{2(2^*-q)\int_{\mathbb{R}^N}|V_1|^q}{q(2^*-2)\int_{\mathbb{R}^N}|V_1|^2}\right)^{\frac{2}{(N-2)(q-2)}}.$$

If $N=4$ or 3. By Fatou's lemma, we have $\|w_0\|_2^2\le\liminf_{\lambda\to0}\|w_\lambda\|_2^2<\infty$, therefore, $w_0=0$ and hence $k=1$. Thus, we obtain $J_0(w^{(1)})=m_0$ and hence $w^{(1)}=V_\rho$ for some $\rho\in(0,+\infty)$. Therefore, we conclude that

$$w_\lambda-\xi_\lambda^{-\frac{N-2}{2}}V_1(\xi_\lambda^{-1}\cdot)\to0$$

in $L^{2^*}(\mathbb{R}^N)$ as $\lambda\to0$, where $\xi_\lambda:=\rho\zeta_\lambda^{(1)}\in(0,+\infty)$ satisfying $\xi_\lambda\to0$ as $\lambda\to0$. Since

$$J_0'(w_\lambda-\xi_\lambda^{-\frac{N-2}{2}}V_1(\xi_\lambda^{-1}\cdot))=J_0'(w_\lambda)+J_0'(V_1)+o(1)=o(1)$$

as $\lambda\to0$, it follows that $w_\lambda-\xi_\lambda^{-\frac{N-2}{2}}V_1(\xi_\lambda^{-1}\cdot)\to0$ in $D^{1,2}(\mathbb{R}^N)$. □

Lemma 5.8. *Let $N\ge5$ and $q\in(2,2^*)$, then there exists a $\zeta_\lambda\in(0,\infty)$ verifying*

$$\zeta_\lambda\sim\lambda^{\frac{2^*-2}{2(q-2)}}$$

such that the rescaled ground states

$$w_\lambda(x)=\zeta_\lambda^{\frac{N-2}{2}}v_\lambda(\zeta_\lambda x)$$

converge to V_{ρ_0} in $H^1(\mathbb{R}^N)$ as $\lambda\to0$, where V_{ρ_0} is given in Lemma 5.7.

Proof. If $\alpha>N-4$, then the statement is valid with $\zeta_\lambda=\lambda^{\frac{2^*-2}{2(q-2)}}$. If $\alpha\le N-4$, then for any $\lambda_n\to0$, up to a subsequence, we can assume that $w_{\lambda_n}\to V_\rho$ in $D^{1,2}(\mathbb{R}^N)$ with $\rho\in(0,\rho_0]$. Moreover, $w_{\lambda_n}\to V_\rho$ in $L^2(\mathbb{R}^N)$ if and only if $\rho=\rho_0$. Arguing as in the proof of Theorem 2.1, we can show that there exists a $\zeta_\lambda\sim\lambda^{\frac{2^*-2}{2(q-2)}}$ such that $w_\lambda(x)=\zeta_\lambda^{\frac{N-2}{2}}v_\lambda(\zeta_\lambda x)$ converges to V_{ρ_0} in $L^2(\mathbb{R}^N)$, and hence in $H^1(\mathbb{R}^N)$. This completes the proof. □

In the lower dimension cases $N=4$ and $N=3$, we further perform a scaling

$$\tilde w(x)=\xi_\lambda^{\frac{N-2}{2}}w(\xi_\lambda x),\tag{5.27}$$

where $\xi_\lambda\in(0,+\infty)$ is given in Lemma 5.7. Then the rescaled equation is as follows

$$-\Delta\tilde w+\lambda^\sigma\xi_\lambda^2\tilde w=(I_\alpha*|\tilde w|^{\frac{N+\alpha}{N-2}})\tilde w^{\frac{2+\alpha}{N-2}}+\lambda^\sigma\xi_\lambda^{N-\frac{N-2}{2}q}\tilde w^{q-1}.\tag{R_λ}$$

The corresponding energy functional is given by

$$\tilde J_\lambda(\tilde w):=\frac12\int_{\mathbb{R}^N}|\nabla\tilde w|^2+\lambda^\sigma\xi_\lambda^2|\tilde w|^2-\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|\tilde w|^p)|\tilde w|^p-\frac1q\lambda^\sigma\xi_\lambda^{N-\frac{N-2}{2}q}\int_{\mathbb{R}^N}|\tilde w|^q.\tag{5.28}$$

Clearly, we have $\tilde{J}_\lambda(\tilde{w}) = J_\lambda(w) = I_\lambda(v)$.

Furthermore, we have the following lemma.

Lemma 5.9. *Let $v, w, \tilde{w} \in H^1(\mathbb{R}^N)$ satisfy (5.1) and (5.27), then the following statements hold true*

(1) $\|\nabla\tilde{w}\|_2^2 = \|\nabla w\|_2^2 = \|\nabla v\|_2^2$, $\int_{\mathbb{R}^N}(I_\alpha * |\tilde{w}|^p)|\tilde{w}|^p = \int_{\mathbb{R}^N}(I_\alpha * |w|^p)|w|^p = \int_{\mathbb{R}^N}(I_\alpha * |v|^p)|v|^p$,

(2) $\xi_\lambda^2\|\tilde{w}\|_2^2 = \|w\|_2^2 = \lambda^{-\sigma}\|v\|_2^2$, $\xi_\lambda^{N-\frac{N-2}{2}q}\|\tilde{w}\|_q^q = \|w\|_q^q = \lambda^{1-\sigma}\|v\|_q^q$.

Set $\tilde{w}_\lambda(x) = \xi_\lambda^{\frac{N-2}{2}} w_\lambda(\xi_\lambda x)$, then by Lemma 5.7, we have

$$\|\nabla(\tilde{w}_\lambda - V_1)\|_2 \to 0, \qquad \|\tilde{w}_\lambda - V_1\|_{2^*} \to 0, \qquad \text{as } \lambda \to 0. \tag{5.29}$$

Note that the corresponding Nehari and Pohožaev's identities are as follows

$$\int_{\mathbb{R}^N} |\nabla\tilde{w}_\lambda|^2 + \lambda^\sigma\xi_\lambda^2\int_{\mathbb{R}^N}|\tilde{w}_\lambda|^2 = \int_{\mathbb{R}^N}(I_\alpha * |\tilde{w}_\lambda|^p)|\tilde{w}_\lambda|^p + \lambda^\sigma\xi_\lambda^{N-\frac{N-2}{2}q}\int_{\mathbb{R}^N}|\tilde{w}_\lambda|^q \tag{5.30}$$

and

$$\frac{1}{2^*}\int_{\mathbb{R}^N} |\nabla\tilde{w}_\lambda|^2 + \frac{1}{2}\lambda^\sigma\xi_\lambda^2\int_{\mathbb{R}^N}|\tilde{w}_\lambda|^2 = \frac{1}{2^*}\int_{\mathbb{R}^N}(I_\alpha * |\tilde{w}_\lambda|^p)|\tilde{w}_\lambda|^p + \frac{1}{q}\lambda^\sigma\xi_\lambda^{N-\frac{N-2}{2}q}\int_{\mathbb{R}^N}|\tilde{w}_\lambda|^q, \tag{5.31}$$

it follows that

$$\left(\frac{1}{2} - \frac{1}{2^*}\right)\lambda^\sigma\xi_\lambda^2\int_{\mathbb{R}^N}|\tilde{w}_\lambda|^2 = \left(\frac{1}{q} - \frac{1}{2^*}\right)\lambda^\sigma\xi_\lambda^{N-\frac{N-2}{2}q}\int_{\mathbb{R}^N}|\tilde{w}_\lambda|^q.$$

Thus, we obtain

$$\xi_\lambda^{\frac{(N-2)(q-2)}{2}}\int_{\mathbb{R}^N}|\tilde{w}_\lambda|^2 = \frac{2(2^*-q)}{q(2^*-2)}\int_{\mathbb{R}^N}|\tilde{w}_\lambda|^q. \tag{5.32}$$

To control the norm $\|\tilde{w}_\lambda\|_2$, we note that for any $\lambda > 0$, $\tilde{w}_\lambda > 0$ satisfies the linear inequality

$$-\Delta\tilde{w}_\lambda + \lambda^\sigma\xi_\lambda^2\tilde{w}_\lambda = (I_\alpha * |\tilde{w}_\lambda|^p)\tilde{w}_\lambda^{p-1} + \lambda^\sigma\xi_\lambda^{N-\frac{N-2}{2}q}\tilde{w}_\lambda^{q-1} > 0, \qquad x \in \mathbb{R}^N. \tag{5.33}$$

Lemma 5.10. *There exists a constant $c > 0$ such that*

$$\tilde{w}_\lambda(x) \ge c|x|^{-(N-2)}\exp(-\lambda^{\frac{\sigma}{2}}\xi_\lambda|x|), \quad |x| \ge 1. \tag{5.34}$$

The proof of the above lemma is similar to that of [36, Lemma 4.8]. As consequences, we have the following lemma.

Lemma 5.11. *If* $N = 3$*, then* $\|\tilde{w}_\lambda\|_2^2 \gtrsim \lambda^{-\frac{\sigma}{2}}\xi_\lambda^{-1}$.

Lemma 5.12. *If* $N = 4$*, then* $\|\tilde{w}_\lambda\|_2^2 \gtrsim -\ln(\lambda^\sigma \xi_\lambda^2)$.

We remark that $\tilde{w}_\lambda$ is only defined for $N = 4$ and $N = 3$. But the following discussion also applies to the case $N \geq 5$. To prove our main result, the key point is to show the boundedness of $\|\tilde{w}_\lambda\|_q$.

Lemma 5.13. *Assume* $N \geq 3$*,* $\alpha > N - 4$ *and* $2 < q < 2^*$*. Then there exist constants* $L_0 > 0$ *and* $C_0 > 0$ *such that for any small* $\lambda > 0$ *and* $|x| \geq L_0\lambda^{-\sigma/2}\xi_\lambda^{-1}$*,*

$$\tilde{w}_\lambda(x) \leq C_0\lambda^{\sigma(N-2)/4}\xi_\lambda^{(N-2)/2}\exp(-\frac{1}{2}\lambda^{\sigma/2}\xi_\lambda|x|).$$

Proof. By (5.25) and (5.26), if $|x| \geq L_0\lambda^{-\sigma/2}\xi_\lambda^{-1}$ with $L_0 > 0$ being large enough, we have

$$\begin{aligned}(I_\alpha * |\tilde{w}_\lambda|^p)(x)|\tilde{w}_\lambda|^{p-2}(x) &= \xi_\lambda^{(N-2)(p-1)-\alpha}(I_\alpha * |w_\lambda|^p)(\xi_\lambda x)|w_\lambda|^{p-2}(\xi_\lambda x) \\ &\leq C\xi_\lambda^2 L_0^{-\frac{N^2-N\alpha+4\alpha}{2(N-2)}}\lambda^{\sigma\cdot\frac{N^2-N\alpha+4\alpha}{4(N-2)}} \\ &\leq \frac{1}{4}\lambda^\sigma\xi_\lambda^2,\end{aligned}$$

here we have used the fact that

$$\frac{N^2 - N\alpha + 4\alpha}{4(N-2)} > 1,$$

which follows from the inequality $N < N + 2 < \frac{4(\alpha+2)}{\alpha-N+4}$, $\forall\alpha \in (N - 4, N)$.

By (5.24) and (5.26), for $|x| \geq L_0\lambda^{-\sigma/2}\xi_\lambda^{-1}$, we get

$$\begin{aligned}\lambda^\sigma\xi_\lambda^{N-\frac{N-2}{2}q}\tilde{w}_\lambda^{q-2}(x) &= \lambda^\sigma\xi_\lambda^{N-\frac{N-2}{2}q}\xi_\lambda^{\frac{N-2}{2}(q-2)}w_\lambda^{q-2}(\xi_\lambda x) \\ &\leq \lambda^\sigma\xi_\lambda^2 \cdot C|\xi_\lambda x|^{-\frac{N}{2}(q-2)} \\ &\leq CL_0^{-N(q-2)/2}\lambda^{\sigma+\frac{\sigma N}{4}(q-2)}\xi_\lambda^2 \\ &\leq \frac{1}{4}\lambda^\sigma\xi_\lambda^2.\end{aligned}$$

Therefore, we obtain

$$-\Delta\tilde{w}_\lambda(x) + \frac{1}{2}\lambda^\sigma\xi_\lambda^2\tilde{w}_\lambda(x) \leq 0, \quad \text{for all } |x| \geq L_0\lambda^{-\sigma/2}\xi_\lambda^{-1}.$$

We adopt an argument as used in [1, Lemma 3.2]. Let $R > L_0\lambda^{-\sigma/2}\xi_\lambda^{-1}$, and introduce a positive function

$$\psi_R(r) := \exp(-\frac{1}{2}\lambda^{\sigma/2}\xi_\lambda(r - L_0\lambda^{-\sigma/2}\xi_\lambda^{-1})) + \exp(\frac{1}{2}\lambda^{\sigma/2}\xi_\lambda(r - R)).$$

It is easy to see that

$$|\psi_R'(r)| \le \frac{1}{2}\lambda^{\sigma/2}\xi_\lambda \psi_R(r), \quad \psi_R''(r) = \frac{1}{4}\lambda^{\sigma}\xi_\lambda^2\psi_R(r).$$

We use the same symbol ψ_R to denote the radial function $\psi_R(|x|)$ on $\mathbb{R}^N$. Then for $L_0\lambda^{-\sigma/2}\xi_\lambda^{-1} < r < R$, if $L_0 \ge 2(N-1)$, then we have

$$\begin{array}{rcl} -\Delta\psi_R + \frac{1}{2}\lambda^\sigma\xi_\lambda^2\psi_R & = & -\psi_R'' - \frac{N-1}{r}\psi_R' + \frac{1}{2}\lambda^\sigma\xi_\lambda^2\psi_R \\ & \ge & -\frac{1}{4}\lambda^\sigma\xi_\lambda^2\psi_R - \frac{N-1}{L_0}\lambda^{\sigma/2}\xi_\lambda \cdot \frac{1}{2}\lambda^{\sigma/2}\xi_\lambda\psi_R + \frac{1}{2}\lambda^\sigma\xi_\lambda^2\psi_R \\ & \ge & 0. \end{array}$$

Furthermore, $\psi_R(L_0\lambda^{-\sigma/2}\xi_\lambda^{-1}) \ge 1$ and $\psi_R(R) \ge 1$, thus we have

$$\tilde{w}_\lambda(R) \le \tilde{w}_\lambda(L_0\lambda^{-\sigma/2}\xi_\lambda^{-1}) \le CL_0^{-\frac{N-2}{2}}\lambda^{\frac{\sigma(N-2)}{4}}\xi_\lambda^{\frac{N-2}{2}}\min\{\psi_R(L_0\lambda^{-\sigma/2}\xi_\lambda^{-1}), \psi_R(R)\}.$$

Hence, the comparison principle implies that if $L_0\lambda^{-\sigma/2}\xi_\lambda^{-1} \le |x| \le R$, then

$$\tilde{w}_\lambda(x) \le CL_0^{-\frac{N-2}{2}}\lambda^{\frac{\sigma(N-2)}{4}}\xi_\lambda^{\frac{N-2}{2}}\psi_R(|x|).$$

Since $R > L_0\lambda^{-\sigma/2}\xi_\lambda^{-1}$ is arbitrary, taking $R \to \infty$, we find that

$$\tilde{w}_\lambda(x) \le CL_0^{-\frac{N-2}{2}}e^{L_0/2}\lambda^{\frac{\sigma(N-2)}{4}}\xi_\lambda^{\frac{N-2}{2}}e^{-\frac{1}{2}\lambda^{\sigma/2}\xi_\lambda|x|},$$

for all $|x| \ge L_0\lambda^{-\sigma/2}\xi_\lambda^{-1}$. The proof is complete. □

In the following proposition, we establish an optimal uniform with respect to λ decay estimate of $\tilde{w}_\lambda$ at infinity.

Proposition 5.14. *Assume $N \ge 3$, $\alpha > N-4$ and $2 < q < 2^*$. Then there exists a constant $C > 0$ such that for small $\lambda > 0$, there holds*

$$\tilde{w}_\lambda(x) \le C(1+|x|)^{-(N-2)}, \qquad x \in \mathbb{R}^N.$$

To prove Proposition 5.14., we first consider the Kelvin transform of $\tilde{w}_\lambda$. For any $w \in H^1(\mathbb{R}^N)$, we denote by $K[w]$ the Kelvin transform of w, that is,

$$K[w](x) := |x|^{-(N-2)}w\left(\frac{x}{|x|^2}\right).$$

It is easy to see that $\|K[\tilde{w}_\lambda]\|_{L^\infty(B_1)} \lesssim 1$ implies that

$$\tilde{w}_\lambda(x) \lesssim |x|^{-(N-2)}, \qquad |x| \ge 1,$$

uniformly for small $\lambda > 0$.

Thus, to prove Proposition 5.14, it needs to show that there exists $\lambda_0 >$ such that

$$\sup_{\lambda\in(0,\lambda_0)} \|K[\tilde{w}_\lambda]\|_{L^\infty(B_1)} < \infty. \tag{5.35}$$

It is easy to verify that $K[\tilde{w}_\lambda]$ satisfies

$$-\Delta K[\tilde{w}_\lambda] + \frac{\lambda^\sigma \xi_\lambda^2}{|x|^4} K[\tilde{w}_\lambda] = \frac{1}{|x|^4}(I_\alpha * |\tilde{w}_\lambda|^p)(\frac{x}{|x|^2})\tilde{w}_\lambda^{p-2}(\frac{x}{|x|^2})K[\tilde{w}_\lambda] + \frac{\lambda^\sigma \xi_\lambda^{\gamma/2}}{|x|^\gamma} K[\tilde{w}_\lambda]^{q-1}, \tag{5.36}$$

here and in what follows, we set

$$\gamma := 2N - (N-2)q > 0.$$

We also see from Lemma 5.13 that if $|x| \le \lambda^{\sigma/2}\xi_\lambda / L_0$, then

$$K[\tilde{w}_\lambda](x) \lesssim \frac{1}{|x|^{N-2}} \lambda^{\frac{\sigma(N-2)}{4}} \xi_\lambda^{\frac{N-2}{2}} e^{-\frac{1}{2}\lambda^{\sigma/2}\xi_\lambda |x|^{-1}}. \tag{5.37}$$

Let

$$a(x) = \frac{\lambda^\sigma \xi_\lambda^2}{|x|^4}, \quad b(x) = \frac{1}{|x|^4}(I_\alpha * |\tilde{w}_\lambda|^p)(\frac{x}{|x|^2})\tilde{w}_\lambda^{p-2}(\frac{x}{|x|^2}) + \frac{\lambda^\sigma \xi^{\gamma/2}}{|x|^\gamma} K[\tilde{w}_\lambda]^{q-2}(x).$$

Then (5.36) reads as

$$-\Delta K[\tilde{w}_\lambda] + a(x)K[\tilde{w}_\lambda] = b(x)K[\tilde{w}_\lambda].$$

We shall apply the Moser iteration to prove (5.35).

We note that it follows from (5.37) that for any $v \in H_0^1(B_4)$,

$$\int_{B_4} \frac{\lambda^\sigma \xi_\lambda^2}{|x|^4} K[\tilde{w}_\lambda](x)|v(x)|dx < \infty. \tag{5.38}$$

Since $\tilde{w}_\lambda \to V_1$ in $L^{2^*}(\mathbb{R}^N)$ as $\lambda \to 0$, and for any $s > 1$, the Lebesgue space $L^s(\mathbb{R}^N)$ has the Kadets-Klee property, it is easy to see that

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^N} |\tilde{w}_\lambda^p - V_1^p|^{\frac{2N}{N+\alpha}} dx = 0$$

and

$$\lim_{\lambda\to 0} \int_{\mathbb{R}^N} |\tilde{w}_\lambda^{p-2} - V_1^{p-2}|^{\frac{2N}{\alpha-N+4}} dx = 0.$$

Therefore, by the Hölder inequality and the Hardy-Littlewood-Sobolev inequality, we find

$$\begin{aligned}
&\left\| \tfrac{1}{|x|^4}(I_\alpha * (|\tilde{w}_\lambda|^p - |V_1|^p))(\tfrac{x}{|x|^2})\tilde{w}_\lambda^{p-2}(\tfrac{x}{|x|^2}) \right\|_{L^{\frac{N}{2}}(\mathbb{R}^N)}^{N/2} \\
&= \int_{\mathbb{R}^N} \tfrac{1}{|x|^{2N}} |(I_\alpha * (|\tilde{w}_\lambda|^p - |V_1|^p))(\tfrac{x}{|x|^2})|^{N/2} |\tilde{w}_\lambda^{p-2}(\tfrac{x}{|x|^2})|^{N/2} dx \\
&= \int_{\mathbb{R}^N} |(I_\alpha * (|\tilde{w}_\lambda|^p - |V_1|^p))(z)|^{N/2} |\tilde{w}_\lambda^{p-2}(z)|^{N/2} dz \\
&\lesssim \left(\int_{\mathbb{R}^N} |\tilde{w}_\lambda^p - V_1^p|^{\frac{2N}{N+\alpha}} \right)^{\frac{N^2-\alpha^2}{4(N-2)}} \left(\int_{\mathbb{R}^N} |\tilde{w}_\lambda|^{2^*} \right)^{\frac{\alpha-N+4}{4}} \\
&\to 0, \quad \text{as } \lambda \to 0
\end{aligned} \tag{5.39}$$

and

$$\begin{aligned}
&\left\| \tfrac{1}{|x|^4}(I_\alpha * |V_1|^p)(\tfrac{x}{|x|^2})[\tilde{w}_\lambda^{p-2}(\tfrac{x}{|x|^2}) - V_1^{p-2}(\tfrac{x}{|x|^2})] \right\|_{L^{\frac{N}{2}}(\mathbb{R}^N)}^{N/2} \\
&= \int_{\mathbb{R}^N} \tfrac{1}{|x|^{2N}} |(I_\alpha * |V_1|^p))(\tfrac{x}{|x|^2})|^{N/2} |\tilde{w}_\lambda^{p-2}(\tfrac{x}{|x|^2}) - V_1^{p-2}(\tfrac{x}{|x|^2})|^{N/2} dx \\
&= \int_{\mathbb{R}^N} |(I_\alpha * |V_1|^p)(z)|^{N/2} |\tilde{w}_\lambda^{p-2}(z) - V_1^{p-2}(z)|^{N/2} dz \\
&\lesssim \left(\int_{\mathbb{R}^N} |V_1|^{2^*} \right)^{\frac{N^2-\alpha^2}{4(N-2)}} \left(\int_{\mathbb{R}^N} |\tilde{w}_\lambda^{p-2}(z) - V_1^{p-2}(z)|^{\frac{2N}{\alpha-N+4}} \right)^{\frac{\alpha-N+4}{4}} \\
&\to 0, \quad \text{as } \lambda \to 0.
\end{aligned} \tag{5.40}$$

It follows from (5.39) and (5.40) that

$$\begin{aligned}
&\left\| \tfrac{1}{|x|^4}(I_\alpha * |\tilde{w}_\lambda|^p)(\tfrac{x}{|x|^2})\tilde{w}_\lambda^{p-2}(\tfrac{x}{|x|^2}) - \tfrac{1}{|x|^4}(I_\alpha * |V_1|^p)(\tfrac{x}{|x|^2})V_1^{p-2}(\tfrac{x}{|x|^2}) \right\|_{L^{\frac{N}{2}}(\mathbb{R}^N)} \\
&\le \left\| \tfrac{1}{|x|^4}(I_\alpha * |V_1|^p)(\tfrac{x}{|x|^2}) \left[\tilde{w}_\lambda^{p-2}(\tfrac{x}{|x|^2}) - V_1^{p-2}(\tfrac{x}{|x|^2}) \right] \right\|_{L^{\frac{N}{2}}(\mathbb{R}^N)} \\
&\quad + \left\| \tfrac{1}{|x|^4}(I_\alpha * (|\tilde{w}_\lambda|^p - |V_1|^p))(\tfrac{x}{|x|^2})\tilde{w}_\lambda^{p-2}(\tfrac{x}{|x|^2}) \right\|_{L^{\frac{N}{2}}(\mathbb{R}^N)} \\
&\quad \to 0, \quad \text{as } \lambda \to 0.
\end{aligned} \tag{5.41}$$

Lemma 5.15. *Assume $N \ge 3$ and $2 < q < 2^*$. Then it holds that*

$$\lim_{\lambda \to 0} \int_{|x| \le 4} \left| \frac{\lambda^\sigma \xi_\lambda^{\gamma/2}}{|x|^\gamma} K[\tilde{w}_\lambda]^{q-2}(x) \right|^{N/2} dx = 0.$$

Proof. We divide the integral into two parts:

$$I_\lambda^{(1)}(\frac{N}{2}) := \int_{|x| \le \lambda^{\sigma/2}\xi_\lambda/L_0} \left| \frac{\lambda^\sigma \xi_\lambda^{\gamma/2}}{|x|^\gamma} K[\tilde{w}_\lambda]^{q-2}(x) \right|^{N/2} dx,$$

$$I_\lambda^{(2)}(\frac{N}{2}) := \int_{\lambda^{\sigma/2}\xi_\lambda/L_0 \le |x| \le 4} \left| \frac{\lambda^\sigma \xi_\lambda^{\gamma/2}}{|x|^\gamma} K[\tilde{w}_\lambda]^{q-2}(x) \right|^{N/2} dx.$$

It follows from (5.37) and the Hölder inequality that

$$\begin{aligned} I_\lambda^{(1)}(\tfrac{N}{2}) &= \lambda^{\frac{\sigma N}{2}}\xi_\lambda^{\frac{\gamma N}{4}}\int_{\lambda^{\sigma/2}|x|\le\xi_\lambda/L_0}|x|^{-\frac{\gamma N}{2}}K[\tilde w_\lambda]^{\frac{N}{2}(q-2)}(x)dx \\ &\lesssim \lambda^{\frac{\sigma N}{8}(4+(N-2)(q-2))}\xi_\lambda^{\frac{N}{4}(\gamma+(N-2)(q-2))} \\ &\quad\cdot\int_{|x|\le\lambda^{\sigma/2}\xi_\lambda/L_0}|x|^{-\frac{\gamma N}{2}-\frac{N(N-2)(q-2)}{2}}e^{-\frac{N(q-2)}{4}\lambda^{\sigma/2}\xi_\lambda|x|^{-1}}dx \\ &= \lambda^{\frac{\sigma N}{8}(8-2\gamma-(N-2)(q-2))}\xi_\lambda^{\frac{N}{4}(4-\gamma-(N-2)(q-2))} \\ &\quad\cdot\int_{L_0}^{+\infty}s^{\frac{\gamma N}{2}-\frac{N(N-2)(q-2)}{2}-N-1}e^{-\frac{N(q-2)}{4}s}ds \\ &\lesssim \lambda^{\frac{\sigma N}{8}(N-2)(q-2)}, \end{aligned}$$

and

$$\begin{aligned} I_\lambda^{(2)}(\tfrac{N}{2}) &= \lambda^{\frac{\sigma N}{2}}\xi_\lambda^{\frac{\gamma N}{4}}\int_{\lambda^{\sigma/2}\xi_\lambda/L_0\le|x|\le4}|x|^{-\frac{\gamma N}{2}}K[\tilde w_\lambda]^{\frac{N}{2}(q-2)}(x)dx \\ &\lesssim \lambda^{\frac{\sigma N}{2}}\xi_\lambda^{\frac{\gamma N}{4}}\left(\int_{\lambda^{\sigma/2}\xi_\lambda/L_0\le|x|\le4}K[\tilde w_\lambda]^{2^*}dx\right)^{1-\frac{\gamma}{4}}\left(\int_{\lambda^{\sigma/2}\xi_\lambda/L_0\le|x|\le4}|x|^{-2N}dx\right)^{\frac{\gamma}{4}} \\ &\lesssim \lambda^{\frac{\sigma N}{2}}\xi_\lambda^{\frac{\gamma N}{4}}\left(\int_{\lambda^{\sigma/2}\xi_\lambda/L_0}^{4}r^{-N-1}dr\right)^{\frac{\gamma}{4}} \\ &\lesssim \lambda^{\frac{\sigma N}{8}(N-2)(q-2)}. \end{aligned}$$

From which the conclusion follows. □

Proof of Proposition 5.14. Since the Kelvin transform is linear and preserves the $D^{1,2}(\mathbb{R}^N)$ norm, it follows from (5.38), (5.41), Lemma 5.15 and Lemma 3.11 (i) that for any $r>1$, there exists $\lambda_r>0$ such that

$$\sup_{\lambda\in(0,\lambda_r)}\|K[\tilde w_\lambda]^r\|_{H^1(B_1)}\le C_r. \tag{5.42}$$

Since $\alpha>N-4$, we have $\frac{2N}{N-\alpha}>\frac{N}{2}$. Firstly, we show that for some $r_0\in(\frac{N}{2},\frac{2N}{N-\alpha})$, there holds

$$\lim_{\lambda\to0}I_\lambda(r_0)=0, \tag{5.43}$$

where

$$I_\lambda(r_0):=\int_{|x|\le4}\left|\frac{1}{|x|^4}\left[(I_\alpha*|\tilde w_\lambda|^p)(\frac{x}{|x|^2})-(I_\alpha*|V_1|^p)(\frac{x}{|x|^2})\right]\tilde w_\lambda^{p-2}(\frac{x}{|x|^2})\right|^{r_0}dx.$$

Since $\alpha>N-4$, for any $r_0\in(\frac{N}{2},\frac{2N}{N-\alpha})$, we have

$$s_1=\frac{2N}{2N-(N-\alpha)r_0}>1,\quad s_2=\frac{2N}{(N-\alpha)r_0}>1,$$

and

$$\frac{1}{s_1}+\frac{1}{s_2}=1.$$

Note that $(N-\alpha)r_0 s_2 = 2N$, by the Hardy-Littlewood-Sobolev inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \tfrac{1}{|x|^{(N-\alpha)r_0 s_2}} |(I_\alpha * (|\tilde{w}_\lambda|^p - |V_1|^p))(\tfrac{x}{|x|^2})|^{r_0 s_2} dx \\
&= \int_{\mathbb{R}^N} |(I_\alpha * (|\tilde{w}_\lambda|^p - |V_1|^p))(z)|^{r_0 s_2} dz \\
&\le C \left(\int_{\mathbb{R}^N} |\tilde{w}_\lambda^p - V_1^p|^{\frac{2N}{(N-2)p}}\right)^{\frac{(N-2)p}{(N-2)p-2\alpha}} \to 0, \quad \text{as } \lambda \to 0.
\end{aligned} \tag{5.44}$$

By the Hölder inequality, we have

$$\begin{aligned}
I_\lambda(r_0) \quad &= \quad \int_{|x|\le 4} \left| \tfrac{1}{|x|^{N-\alpha}} (I_\alpha * (|\tilde{w}_\lambda|^p - |V_1|^p))(\tfrac{x}{|x|^2}) K[\tilde{w}_\lambda]^{p-2}(x) \right|^{r_0} dx \\
&= \quad \left(\int_{|x|\le 4} K[\tilde{w}_\lambda]^{(p-2)r_0 s_1}\right)^{\frac{1}{s_1}} \\
&\quad \cdot \left(\int_{\mathbb{R}^N} |x|^{-(N-\alpha)r_0 s_2} \left| (I_\alpha * (|\tilde{w}_\lambda|^p - |V_1|^p))(\tfrac{x}{|x|^2}) \right|^{r_0 s_2} dx\right)^{\frac{1}{s_2}},
\end{aligned}$$

which together with (5.42) and (5.44) yields (5.43).

Next, we consider the function

$$J_\lambda(r_0) := \int\limits_{|x|\le 4} \left| \frac{1}{|x|^4} (I_\alpha * |V_1|^p)(\frac{x}{|x|^2}) \tilde{w}_\lambda^{p-2}(\frac{x}{|x|^2}) \right|^{r_0} dx.$$

Then it follows from the Hölder inequality that

$$\begin{aligned}
J_\lambda(r_0) \quad &= \quad \int_{\frac{1}{4}\le |z| <\infty} |z|^{4r_0 - 2N} \left| (I_\alpha * |V_1|^p)(z) \tilde{w}_\lambda^{p-2}(z) \right|^{r_0} dz \\
&\le \quad \left(\int_{\frac{1}{4}\le |z| <\infty} |z|^{(4r_0-2N)s_1} |(I_\alpha * |V_1|^p)(z)|^{r_0 s_1} dz\right)^{\frac{1}{s_1}} \\
&\quad \cdot \left(\int_{\mathbb{R}^N} |\tilde{w}_\lambda^{p-2}(z)|^{r_0 s_2} dz\right)^{\frac{1}{s_2}},
\end{aligned} \tag{5.45}$$

where

$$s_1 = \frac{2N}{2N-(\alpha - N + 4)r_0}, \qquad s_2 = \frac{2N}{(\alpha - N + 4)r_0}.$$

Since $p = \frac{N+\alpha}{N-2} > \frac{N}{N-2}$, we have

$$\frac{N}{2} < \frac{2N}{\alpha - N + 4} = \frac{2N}{(N-2)(p-2)}.$$

Consider the function

$$h(r_0) := (4r_0 - 2N)s_1 - (N-\alpha)r_0 s_1 + N.$$

It is easy to check that $h(\frac{N}{2}) = -N < 0$, and hence $h(r_0) < 0$ for $r_0 > \frac{N}{2}$ which is close to $\frac{N}{2}$.

Since $p > \frac{N}{N-2}$, we have $\int_{\mathbb{R}^N} |V_1|^p < \infty$. Notice that

$$|V_1|^p |x|^N \le |x|^{-(N-2)p+N} \to 0$$

as $|x| \to \infty$, by Lemma 3.10, we have

$$\begin{aligned}&\int_{\frac{1}{4}\le|z|<\infty} |z|^{(4r_0-2N)s_1} |(I_\alpha * |V_1|^p)(z)|^{r_0 s_1} dz \\ &\lesssim \int_{\frac{1}{4}\le|z|<\infty} |z|^{(4r_0-2N)s_1} |z|^{-(N-\alpha)r_0 s_1} dz \\ &= \int_{\frac{1}{4}}^{\infty} r^{(4r_0-2N)s_1-(N-\alpha)r_0 s_1+N-1} dr < \infty.\end{aligned} \tag{5.46}$$

On the other hand, noting that $r_0 s_2 = \frac{2N}{(N-2)(q-2)}$, we have

$$\int_{\mathbb{R}^N} |\tilde{w}_\lambda^{p-2}(z)|^{r_0 s_2} dz = \int_{\mathbb{R}^N} |\tilde{w}_\lambda|^{2^*} = \int_{\mathbb{R}^N} |w_\lambda|^{2^*} < C < \infty. \tag{5.47}$$

Thus, from (5.45), (5.46) and (5.47), it follows that there is $\lambda_0 > 0$ such that

$$\sup_{\lambda\in(0,\lambda_0)} J_\lambda(r_0) < +\infty,$$

which together with (5.43) implies that for some $\lambda_0 > 0$, there holds

$$\sup_{\lambda\in(0,\lambda_0)} \int_{|x|\le 4} \left| \frac{1}{|x|^4} (I_\alpha * |\tilde{w}_\lambda|^p)(\frac{x}{|x|^2}) \tilde{w}_\lambda^{p-2}(\frac{x}{|x|^2}) \right|^{r_0} dx < +\infty.$$

It remains to prove that there exists $r_0 > \frac{N}{2}$ and $\lambda_0 > 0$ such that

$$\sup_{\lambda\in(0,\lambda_0)} \int_{|x|\le 4} \left| \frac{\lambda^\sigma \xi_\lambda^{\gamma/2}}{|x|^\gamma} K[\tilde{w}_\lambda]^{q-2}(x) \right|^{r_0} dx \le C_{r_0}. \tag{5.48}$$

It is easy to see that $0 < \gamma < 4$. Put $\eta_\lambda = \lambda^{\sigma/2} \xi_\lambda / L_0$. Let $\theta \in (0, 1)$ and $s_0 > 1$, then by (5.42), (5.37) and the Hölder inequality, we have

$$\begin{aligned} I_\lambda^{(1)}(r_0) :&= \lambda^{\sigma r_0} \xi_\lambda^{\gamma r_0/2} \left(\int_{|x|\le\eta_\lambda} \frac{1}{|x|^{\gamma r_0 s_0}} K[\tilde{w}_\lambda]^{(q-2)\theta r_0 s_0}(x) dx \right)^{\frac{1}{s_0}} \\ &\quad \cdot \left(\int_{|x|\le\eta_\lambda} K[\tilde{w}_\lambda]^{\frac{(q-2)(1-\theta)r_0 s_0}{s_0-1}}(x) dx \right)^{1-\frac{1}{s_0}} \\ &\lesssim \lambda^{\sigma r_0 + \frac{\sigma(N-2)(q-2)\theta}{4} r_0} \xi_\lambda^{\frac{\gamma}{2} r_0 + \frac{(N-2)(q-2)\theta}{2} r_0} \\ &\quad \cdot \left(\int_{|x|\le\eta_\lambda} |x|^{-\gamma r_0 s_0 - (N-2)(q-2)\theta r_0 s_0} e^{-\frac{1}{2}(q-2) r_0 \lambda^{\sigma/2} \xi_\lambda |x|^{-1}} dx \right)^{\frac{1}{s_0}} \\ &\lesssim \lambda^{\Gamma_1(r_0,s_0,\theta)} \xi_\lambda^{\Gamma_2(r_0,s_0,\theta)} \left(\int_{L_0}^{+\infty} t^{\gamma r_0 s_0 + (N-2)(q-2)\theta r_0 s_0 - N-1} e^{-\frac{1}{2}(q-2)\theta r_0 s_0 t} dt \right)^{\frac{1}{s_0}}, \end{aligned}$$

where

$$\Gamma_1(r_0, s_0, \theta) = \sigma r_0 + \frac{\sigma(N-2)(q-2)\theta r_0}{4} + \frac{\sigma}{2}[-\gamma r_0 - (N-2)(q-2)\theta r_0 + \frac{N}{s_0}],$$
$$\Gamma_2(r_0, s_0, \theta) = \frac{\sigma r_0}{2} + \frac{\sigma(N-2)(q-2)\theta r_0}{2} - \gamma r_0 - (N-2)(q-2)\theta r_0 + \frac{N}{s_0}.$$

Since $\Gamma_1(\frac{N}{2}, 1, 0) = \frac{\sigma N}{4}(4-\gamma) > 0$ and $\Gamma_2(\frac{N}{2}, 1, 0) = \frac{N}{2}(4-\gamma) > 0$, we choose $r_0 > \frac{N}{2}$, $s_0 > 1$ and $\theta > 0$ such that $\Gamma_1(r_0, s_0, \theta) > 0$ and $\Gamma_2(r_0, s_0, \theta) > 0$. Therefore, we obtain

$$\lim_{\lambda\to 0} I_\lambda^{(1)}(r_0) = 0. \tag{5.49}$$

By (5.42) and the Hölder inequality, we also have

$$\begin{aligned}
I_\lambda^{(2)}(r_0) : &= \lambda^{\sigma r_0}\xi_\lambda^{\gamma r_0/2}\int_{\eta_\lambda\le|x|\le 4}\left|\frac{1}{|x|^\gamma}K[\tilde{w}_\lambda]^{(q-2)}(x)\right|^{r_0}dx \\
&\le \lambda^{\sigma r_0}\xi_\lambda^{\gamma r_0/2}\left(\int_{|x|\le 4}K[\tilde{w}_\lambda]^{\frac{(q-2)r_0 s_0}{s_0-1}}\right)^{1-\frac{1}{s_0}}\left(\int_{\eta_\lambda\le|x|\le 4}|x|^{-\gamma r_0 s_0}dx\right)^{\frac{1}{s_0}} \\
&\lesssim \lambda^{\sigma r_0}\xi_\lambda^{\gamma r_0/2}\left(\int_{\eta_\lambda}^4 r^{-\gamma r_0 s_0+N-1}dr\right)^{\frac{1}{s_0}} \\
&\lesssim \lambda^{\sigma r_0}\xi_\lambda^{\gamma r_0/2}\eta_\lambda^{-\frac{\gamma r_0 s_0-N}{s_0}} = \lambda^{\Gamma_3(r_0,s_0)}\xi_\lambda^{\Gamma_4(r_0,s_0)},
\end{aligned}$$

where $\Gamma_3(r_0, s_0) = \frac{\sigma}{2s_0}[N-(\gamma-2)r_0 s_0]$, and $\Gamma_4(r_0, s_0) = \frac{1}{2s_0}[2N-\gamma r_0 s_0]$. Since $\Gamma(\frac{N}{2}, 1) = \frac{\sigma N}{4}(4-\gamma) > 0$ and $\Gamma_4(\frac{N}{2}, 1) = \frac{N}{4}(4-\gamma) > 0$, we choose $r_0 > \frac{N}{2}$, $s_0 > 1$ such that $\Gamma_3(r_0, s_0) > 0$ and $\Gamma_4(r_0, s_0) > 0$. Therefore, we obtain

$$\lim_{\lambda\to 0} I_\lambda^{(2)}(r_0) = 0. \tag{5.50}$$

Finally, (5.48) follows from (5.49) and (5.50), thus, by Lemma 3.11 (ii), it follows that (5.35) holds, and hence

$$\tilde{w}_\lambda(x) \lesssim |x|^{-(N-2)}, \qquad |x| \ge 1,$$

uniformly for small $\lambda > 0$.

Let $\hat{a}(x) = \lambda^\sigma \xi_\lambda^2$ and

$$\hat{b}(x) = (I_\alpha * |\tilde{w}_\lambda|^p)\tilde{w}_\lambda^{p-2} + \lambda^\sigma \xi_\lambda^{N-\frac{N-2}{2}q}\tilde{w}_\lambda^{q-2}.$$

Then

$$-\Delta\tilde{w}_\lambda + \hat{a}(x)\tilde{w}_\lambda = \hat{b}(x)\tilde{w}_\lambda.$$

Applying the Moser iteration again, it is easy to show that there exists a constant $\hat{\lambda}_0 > 0$ such that

$$\sup_{\lambda\in(0,\hat{\lambda}_0)} \|\tilde{w}_\lambda\|_{L^\infty(B_1)} \le C < \infty.$$

Thus, we conclude that $\tilde{w}_\lambda(x) \lesssim (1+|x|)^{-(N-2)}$. The proof of Proposition 5.14 is complete. □

Lemma 5.16. *If $q > \frac{N}{N-2}$, then $\|\tilde{w}_\lambda\|_q^q \sim 1$ as $\lambda \to 0$. Furthermore, $\tilde{w}_\lambda \to V_1$ in $L^q(\mathbb{R}^N)$ as $\lambda \to 0$.*

Proof. Since $\tilde{w}_\lambda \to V_1$ in $L^{2^*}(\mathbb{R}^N)$, as in [36, Lemma 4.6], using the embeddings $L^{2^*}(B_1) \hookrightarrow L^q(B_1)$ we prove that $\liminf_{\lambda\to 0} \|w_\lambda\|_q^q > 0$.

On the other hand, by virtue of Proposition 5.14, there exists a constant $C > 0$ such that for all small $\lambda > 0$,

$$\tilde{w}_\lambda(x) \le \frac{C}{(1+|x|)^{N-2}}, \qquad \forall x \in \mathbb{R}^N,$$

which together with the fact that $q > \frac{N}{N-2}$ implies that $\tilde{w}_\lambda$ is bounded in $L^q(\mathbb{R}^N)$ uniformly for small $\lambda > 0$, and by the dominated convergence theorem $\tilde{w}_\lambda \to V_1$ in $L^q(\mathbb{R}^N)$ as $\lambda \to 0$. □

Proof of Theorem 2.2. For $N \ge 5$, the conclusion follows directly from Lemmas 5.5, 5.6 and 5.8. We only consider the cases $N = 4$ and $N = 3$.

We first note that a result similar to Lemma 3.2 holds for $\tilde{w}_\lambda$ and $\tilde{J}_\lambda$. By Lemma 5.9, we also have $\tau_2(\tilde{w}_\lambda) = \tau_2(w_\lambda)$. Therefore, by (5.32), we get

$$\begin{aligned} m_0 &\le \sup_{t\ge 0} \tilde{J}_\lambda((\tilde{w}_\lambda)_t) + \lambda^\sigma \tau_2(\tilde{w}_\lambda)^{\frac{N}{2}} \left\{ \frac{1}{q} \xi_\lambda^{N-\frac{N-2}{2}q} \int_{\mathbb{R}^N} |\tilde{w}_\lambda|^q - \frac{1}{2}\xi_\lambda^2 \int_{\mathbb{R}^N} |\tilde{w}_\lambda|^2 \right\} \\ &= m_\lambda + \lambda^\sigma \tau_2(\tilde{w}_\lambda)^{\frac{N}{2}} \frac{q-2}{q(2^*-2)} \xi_\lambda^{N-\frac{N-2}{2}q} \int_{\mathbb{R}^N} |\tilde{w}_\lambda|^q, \end{aligned} \tag{5.51}$$

which implies that

$$\xi_\lambda^{N-\frac{N-2}{2}q} \int_{\mathbb{R}^N} |\tilde{w}_\lambda|^q \ge \lambda^{-\sigma} \frac{q(2^*-2)}{(q-2)\tau_2(w_\lambda)^{\frac{N}{2}}} \delta_\lambda.$$

Hence, by Lemma 5.5, we obtain

$$\xi_\lambda^{N-\frac{N-2}{2}q} \int_{\mathbb{R}^N} |\tilde{w}_\lambda|^q \gtrsim \lambda^{-\sigma}\delta_\lambda \gtrsim \begin{cases} (\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}, & \text{if} \quad N=4, \\ \lambda^{\frac{2(6-q)}{(q-2)(q-4)}}, & \text{if} \quad N=3. \end{cases} \tag{5.52}$$

Therefore, by Lemma 5.16, we have

$$\xi_\lambda \gtrsim \begin{cases} (\ln\frac{1}{\lambda})^{-\frac{1}{q-2}}, & \text{if} \quad N=4, \\ \lambda^{\frac{4}{(q-2)(q-4)}}, & \text{if} \quad N=3. \end{cases} \tag{5.53}$$

On the other hand, if $N = 3$, then by (5.32) and Lemma 5.11 and Lemma 5.16, we have

$$\xi_\lambda^{\frac{q-2}{2}} \lesssim \frac{1}{\|\tilde{w}_\lambda\|_2^2} \lesssim \lambda^{\frac{\sigma}{2}} \xi_\lambda.$$

Then

$$\xi_\lambda^{\frac{q-4}{2}} \lesssim \lambda^{\frac{\sigma}{2}}.$$

Hence, noting that $\sigma = \frac{2^*-2}{q-2} = \frac{4}{q-2}$, for $q \in (4,6)$, we have

$$\xi_\lambda \lesssim \lambda^{\frac{4}{(q-2)(q-4)}}. \tag{5.54}$$

If $N=4$, then by (5.32) and Lemma 5.12 and Lemma 5.16, we have

$$\xi_\lambda^{q-2} \lesssim \frac{1}{\|\tilde{w}_\lambda\|_2^2} \lesssim \frac{1}{-\ln(\lambda^\sigma \xi_\lambda^2)}.$$

Note that

$$-\ln(\lambda^\sigma \xi_\lambda^2) = \sigma \ln\frac{1}{\lambda} + 2\ln\frac{1}{\xi_\lambda} \ge \sigma \ln\frac{1}{\lambda},$$

it follows that

$$\xi_\lambda^{q-2} \lesssim \frac{1}{\|\tilde{w}_\lambda\|_2^2} \lesssim \left(\ln\frac{1}{\lambda}\right)^{-1}.$$

Hence, we obtain

$$\xi_\lambda \lesssim \left(\ln\frac{1}{\lambda}\right)^{-\frac{1}{q-2}}. \tag{5.55}$$

Thus, it follows from (5.51), (5.54), (5.55) and Lemma 5.16 that

$$\delta_\lambda = m_0 - m_\lambda \lesssim \lambda^\sigma \xi_\lambda^{N-\frac{N-2}{2}q} \lesssim \begin{cases} \lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}, & \text{if} \quad N=4, \\ \lambda^{\frac{2}{q-4}}, & \text{if} \quad N=3, \end{cases}$$

which together with Lemma 5.5 implies that

$$\delta_\lambda \sim \lambda^\sigma \xi_\lambda^{N-\frac{N-2}{2}q} \sim \begin{cases} \lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}, & \text{if} \quad N=4, \\ \lambda^{\frac{2}{q-4}}, & \text{if} \quad N=3. \end{cases} \tag{5.56}$$

By (5.28) and (5.32), we get

$$m_\lambda = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \tilde{w}_\lambda|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |\tilde{w}_\lambda|^p)|\tilde{w}_\lambda^p - \frac{q-2}{q(2^*-2)}\lambda^\sigma \xi_\lambda^{N-\frac{N-2}{2}q}\int_{\mathbb{R}^N} |\tilde{w}_\lambda|^q.$$

By (5.30) and (5.31), we get

$$\int_{\mathbb{R}^N} |\nabla \tilde{w}_\lambda|^2 = \int_{\mathbb{R}^N} (I_\alpha * |\tilde{w}_\lambda|^p)|\tilde{w}_\lambda|^p + \frac{N(q-2)}{2q}\lambda^\sigma \xi_\lambda^{N-\frac{N-2}{2}q} \int_{\mathbb{R}^N} |\tilde{w}_\lambda|^q.$$

Therefore, we have

$$m_\lambda = \frac{2+\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla \tilde{w}_\lambda|^2 - \frac{\alpha(N-2)(q-2)}{4q(N+\alpha)}\lambda^\sigma \xi_\lambda^{N-\frac{N-2}{2}q} \int_{\mathbb{R}^N} |\tilde{w}_\lambda|^q.$$

Similarly, we have

$$m_0 = \frac{2+\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla V_1|^2.$$

Thus, by virtue of (5.56), we obtain

$$\begin{aligned} \|\nabla V_1\|_2^2 - \|\nabla \tilde{w}_\lambda\|_2^2 &= \tfrac{2(N+\alpha)}{2+\alpha}\delta_\lambda - \tfrac{\alpha(N-2)(q-2)}{2q(2+\alpha)}\lambda^\sigma \xi_\lambda^{N-\frac{N-2}{2}q} \textstyle\int_{\mathbb{R}^N} |\tilde{w}_\lambda|^q \\ &= \begin{cases} O(\lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}), & \text{if} \quad N=4, \\ O(\lambda^{\frac{2}{q-4}}), & \text{if} \quad N=3. \end{cases} \end{aligned} \tag{5.57}$$

By (5.28) and (5.31), we have

$$\begin{aligned} m_\lambda &= (\tfrac{1}{2} - \tfrac{1}{2^*})\textstyle\int_{\mathbb{R}^N} |\nabla \tilde{w}_\lambda|^2 + (\tfrac{1}{2^*} - \tfrac{1}{2p})\int_{\mathbb{R}^N} (I_\alpha * |\tilde{w}_\lambda|^p)|\tilde{w}_\lambda|^p \\ &= \tfrac{1}{N}\textstyle\int_{\mathbb{R}^N} |\nabla \tilde{w}_\lambda|^2 + \tfrac{\alpha(N-2)}{2N(N+\alpha)}\int_{\mathbb{R}^N} (I_\alpha * |\tilde{w}_\lambda|^p)|\tilde{w}_\lambda|^p. \end{aligned}$$

Similarly, we also have

$$m_0 = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla V_1|^2 + \frac{\alpha(N-2)}{2N(N+\alpha)}\int_{\mathbb{R}^N} (I_\alpha * |V_1|^p)|V_1|^p.$$

Then it follows from (5.56) and (5.57) that

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^N} (I_\alpha * |V_1|^p)|V_1|^p - \int_{\mathbb{R}^N} (I_\alpha * |\tilde{w}_\lambda|^p)|\tilde{w}_\lambda|^p \\ &= \tfrac{2N(N+\alpha)}{\alpha(N-2)}\left[(m_0 - m_\lambda) - \tfrac{1}{N}(\textstyle\int_{\mathbb{R}^N} |\nabla V_1|^2 - \int_{\mathbb{R}^N} |\nabla \tilde{w}_\lambda|^2)\right] \\ &= \begin{cases} O(\lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}), & \text{if} \quad N=4, \\ O(\lambda^{\frac{2}{q-4}}), & \text{if} \quad N=3. \end{cases} \end{aligned} \tag{5.58}$$

Since $\|\nabla V_1\|_2^2 = \int_{\mathbb{R}^N} (I_\alpha * |V_1|^p)|V_1|^p = S_\alpha^{\frac{N+\alpha}{2+\alpha}}$, it follows from (5.57) and (5.58) that

$$\|\nabla \tilde{w}_\lambda\|_2^2 = S_\alpha^{\frac{N+\alpha}{2+\alpha}} + \begin{cases} O(\lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}), & \text{if} \quad N=4, \\ O(\lambda^{\frac{2}{q-4}}), & \text{if} \quad N=3, \end{cases}$$

and

$$\int\limits_{\mathbb{R}^N} (I_\alpha * |\tilde{w}_\lambda|^p)|\tilde{w}_\lambda|^p = S_\alpha^{\frac{N+\alpha}{2+\alpha}} + \begin{cases} O(\lambda^{\frac{2}{q-2}}(\ln\frac{1}{\lambda})^{-\frac{4-q}{q-2}}), & \text{if} \quad N=4, \\ O(\lambda^{\frac{2}{q-4}}), & \text{if} \quad N=3. \end{cases}$$

Finally, by (5.32), Lemma 5.11 and Lemma 5.12, we obtain

$$\|\tilde{w}_\lambda\|_2^2 \sim \begin{cases} \ln\frac{1}{\lambda}, & \text{if } N=4, \\ \lambda^{-\frac{2}{q-4}}, & \text{if } N=3. \end{cases}$$

The statements on v_λ follow from the corresponding results on w_λ and $\tilde{w}_\lambda$. This completes the proof of Theorem 2.2. □

6. Proofs of other results and final remarks

In this section first we prove Theorem 2.3 and Theorem 2.4. We consider (Q_λ) and its limit equation

$$-\Delta v + v = (I_\alpha * |v|^p)|v|^{p-2}v. \tag{6.1}$$

The corresponding energies of ground states are given by $m_\lambda = \inf_{v\in\mathcal{M}_\lambda} I_\lambda(v)$ and

$$m_0 := \inf_{v\in\mathcal{M}_0} I_0(v),$$

where

$$I_0(v) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla v|^2 + |v|^2 - \frac{1}{2p}\int\limits_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p, \tag{6.2}$$

and

$$\mathcal{M}_0 = \left\{ v \in H^1(\mathbb{R}^N)\setminus\{0\} \;\middle|\; \int\limits_{\mathbb{R}^N} |v|^2 + |v|^2 = \int\limits_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p \right\}.$$

Then m_λ and m_0 are well-defined and positive. Moreover, I_0 is attained on $\mathcal{M}_0$ by positive solutions of (6.1).

Lemma 6.1. *Assume that the assumptions of Theorem 2.3 hold. Then for small $\lambda > 0$, there holds*

$$m_0 - m_\lambda \sim \lambda.$$

Proof. The proof is similar to that of Lemma 4.5 and is omitted. □

Proof of Theorem 2.3. Since v_λ is bounded in $H^1(\mathbb{R}^N)$, there exists $v_0 \in H^1(\mathbb{R}^N)$ verifying $-\Delta v + v = (I_\alpha * |v|^p)|v|^{p-2}v$ such that up to a subsequence, we have

$$v_\lambda \rightharpoonup v_0 \quad \text{weakly in } H^1(\mathbb{R}^N), \quad v_\lambda \to v_0 \quad \text{in } L^p(\mathbb{R}^N) \quad \text{for any } p \in (2, 2^*),$$

and

$$v_\lambda(x) \to v_0(x) \quad a.\, e. \text{ on } \mathbb{R}^N, \qquad v_\lambda \to v_0 \quad \text{in } L^2_{loc}(\mathbb{R}^N).$$

Being a ground state solution, v_λ satisfies

$$m_\lambda = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + |v_\lambda|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^p)|v_\lambda|^p - \frac{1}{q}\lambda \int_{\mathbb{R}^N} |v_\lambda|^q. \tag{6.3}$$

Since $v_\lambda \in \mathcal{M}_\lambda$, we also have

$$\int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + |v_\lambda|^2 = \int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^p)|v_\lambda|^p + \lambda \int_{\mathbb{R}^N} |v_\lambda|^q. \tag{6.4}$$

Furthermore, by Lemma 3.1, the Pohožaev identity is given by

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + \frac{N}{2}\int_{\mathbb{R}^N} |v_\lambda|^2 = \frac{N+\alpha}{2p}\int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^p)|v_\lambda|^p + \frac{N}{q}\lambda \int_{\mathbb{R}^N} |v_\lambda|^q. \tag{6.5}$$

Let

$$A_\lambda = \int_{\mathbb{R}^N} |\nabla v_\lambda|^2,\ B_\lambda = \int_{\mathbb{R}^N} |v_\lambda|^2,\ C_\lambda = \int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^p)|v_\lambda|^p,\ D_\lambda = \int_{\mathbb{R}^N} |v_\lambda|^q.$$

Then by (6.4) and (6.5), we have

$$\begin{cases} A_\lambda - C_\lambda = -B_\lambda + \lambda D_\lambda, \\ \frac{N-2}{2}A_\lambda - \frac{N+\alpha}{2p}C_\lambda = -\frac{N}{2}B_\lambda + \frac{N}{q}\lambda D_\lambda. \end{cases}$$

Solving this system to obtain

$$A_\lambda = \frac{1}{\eta}\left[(N(p-1)-\alpha)B_\lambda + (N+\alpha - \frac{2Np}{q})\lambda D_\lambda\right],$$

$$C_\lambda = \frac{1}{\eta}\left[2pB_\lambda + (p(N-2) - \frac{2Np}{q})\lambda D_\lambda\right],$$

where $\eta = N + \alpha - p(N-2) > 0$. Therefore, we obtain

$$\begin{aligned} m_\lambda &= \tfrac{1}{2}A_\lambda + \tfrac{1}{2}B_\lambda - \tfrac{1}{2p}C_\lambda - \tfrac{1}{q}\lambda D_\lambda \\ &= \tfrac{p-1}{\eta}B_\lambda + \tfrac{q(2+\alpha)-2(2p+\alpha)}{2\eta q}\lambda D_\lambda \\ &= \tfrac{p-1}{\eta}\int_{\mathbb{R}^N}|v_\lambda|^2 + \tfrac{q(2+\alpha)-2(2p+\alpha)}{2\eta q}\lambda\int_{\mathbb{R}^N}|v_\lambda|^q. \end{aligned}$$

Moreover, the ground state v_0 of (6.1) also satisfies the following identities:

$$m_0 = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_0|^2 + |v_0|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |v_0|^p)|v_0|^p, \tag{6.6}$$

$$\int_{\mathbb{R}^N}|\nabla v_0|^2 + |v_0|^2 = \int_{\mathbb{R}^N}(I_\alpha * |v_0|^p)|v_0|^p, \tag{6.7}$$

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v_0|^2 + \frac{N}{2}\int_{\mathbb{R}^N}|v_0|^2 = \frac{N+\alpha}{2p}\int_{\mathbb{R}^N}(I_\alpha * |v_0|^p)|v_0|^p. \tag{6.8}$$

In a similar way, we show that

$$m_0 = \frac{p-1}{\eta}\int_{\mathbb{R}^N}|v_0|^2.$$

Therefore,

$$\frac{p-1}{\eta}\int_{\mathbb{R}^N}|v_\lambda|^2 - |v_0|^2 = m_\lambda - m_0 - \frac{q(2+\alpha)-2(2p+\alpha)}{2\eta q}\lambda\int_{\mathbb{R}^N}|v_\lambda|^q,$$

which together with Lemma 6.1 and $q \geq \frac{2(2p+\alpha)}{2+\alpha}$ implies that

$$\|v_0\|_2^2 - \|v_\lambda\|_2^2 \sim \lambda.$$

Arguing in a similar way, we show that

$$\frac{p-1}{Np-(N+\alpha)}\int_{\mathbb{R}^N}|\nabla v_\lambda|^2 - |\nabla v_0|^2 = m_\lambda - m_0 + \frac{\alpha(q-2)}{2q[Np-(N+\alpha)]}\lambda\int_{\mathbb{R}^N}|v_\lambda|^q.$$

Therefore, we obtain

$$\|\nabla v_\lambda\|_2^2 - \|\nabla v_0\|_2^2 = O(\lambda).$$

We have proved $\|\nabla v_\lambda\|_2 \to \|\nabla v_0\|_2$ and $\|v_\lambda\|_2 \to \|v_0\|_2$, therefore, we obtain $v_\lambda \to v_0$ in $H^1(\mathbb{R}^N)$. □

Now, we consider (Q_μ) and its limit equation

$$-\Delta v + v = |v|^{p-2}v. \tag{6.9}$$

The corresponding energies of ground states are given by $m_\mu = \inf_{v\in\mathcal{M}_\mu} I_\mu(v)$ and

$$m_0 := \inf_{v\in\mathcal{M}_0} I_0(v),$$

where

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2 - \frac{1}{q}\int_{\mathbb{R}^N} |v|^q, \tag{6.10}$$

and

$$\mathcal{M}_0 = \left\{ v \in H^1(\mathbb{R}^N)\setminus\{0\} \,\middle|\, \int_{\mathbb{R}^N} |v|^2 + |v|^2 = \int_{\mathbb{R}^N} |v|^q \right\}.$$

Then m_μ and m_0 are well-defined and positive. Moreover, I_0 is attained on $\mathcal{M}_0$ by the unique positive solution of (6.9).

Lemma 6.2. *Assume that the assumptions of Theorem 2.4 hold. Then for small $\mu > 0$, there holds*

$$m_0 - m_\mu \sim \mu.$$

Proof. The proof is similar to that of Lemma 4.5 and is omitted. □

Proof of Theorem 2.4. Arguing as in the proof of Theorem 2.3, Theorem 2.4 follows from Lemma 6.2 and the details will be omitted. □

Finally, we consider the case $\lambda \to \infty$ in (Q_λ) and $\mu \to \infty$ in (Q_μ). It is easy to see that under the rescaling

$$w(x) = \lambda^{\frac{1}{q-2}} v(x),$$

the equation (Q_λ) is reduced to

$$-\Delta w + w = \lambda^{-\frac{2(p-1)}{q-2}} (I_\alpha * |w|^p)|w|^{p-2}w + |w|^{q-2}w.$$

Therefore, by Theorem 2.4, we have the following.

Theorem 6.1. *Let $N \geq 3$, $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$, $q \in (2, 2^*)$ and v_λ be the ground state of (Q_λ), then as $\lambda \to \infty$, the rescaled family of ground states $\tilde{w}_\lambda = \lambda^{\frac{1}{q-2}} v_\lambda$ converges in $H^1(\mathbb{R}^N)$ to the unique positive solution $w_0 \in H^1(\mathbb{R}^N)$ of the equation*

$$-\Delta w + w = w^{q-1}.$$

Moreover, as $\lambda \to \infty$, there holds

$$\|v_\lambda\|_2^2 = \begin{cases} \lambda^{-\frac{2}{q-2}} \left(\frac{2N-q(N-2)}{2q} S_q^{\frac{q}{q-2}} + O(\lambda^{-\frac{2(p-1)}{q-2}}) \right), & \text{if} \quad q > \frac{2(2p+\alpha)}{q-2}, \\ \lambda^{-\frac{2}{q-2}} \left(\frac{2N-q(N-2)}{2q} S_q^{\frac{q}{q-2}} - \Theta(\lambda^{-\frac{2(p-1)}{q-2}}) \right), & \text{if} \quad q \leq \frac{2(2p+\alpha)}{q-2}, \end{cases}$$

$$\|\nabla v_\lambda\|_2^2 = \lambda^{-\frac{2}{q-2}} \left(\frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} + O(\lambda^{-\frac{2(p-1)}{q-2}}) \right),$$

and the least energy m_λ of the ground state satisfies

$$\frac{q-2}{2q} S_q^{\frac{q}{q-2}} - \lambda^{\frac{2}{q-2}} m_\lambda \sim \lambda^{-\frac{2(p-1)}{q-2}},$$

as $\lambda \to \infty$, where S_q is given in (2.14).

Under the rescaling

$$w(x) = \mu^{\frac{1}{2(p-1)}} v(x),$$

the equation (Q_μ) is reduced to

$$-\Delta w + w = (I_\alpha * |w|^p)|w|^{p-2} w + \mu^{-\frac{q-2}{2(p-1)}} |w|^{q-2} w.$$

Then by Theorem 2.3 we have the following.

Theorem 6.2. *Let $N \geq 3$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$, $q \in (2, 2^*]$ and v_μ be the ground state of (Q_μ), then as $\mu \to \infty$, the rescaled family of ground states $\tilde{w}_\mu = \mu^{\frac{1}{2(p-1)}} v_\mu$ converges up to a subsequence in $H^1(\mathbb{R}^N)$ to a positive solution $w_0 \in H^1(\mathbb{R}^N)$ of the equation*

$$-\Delta v + v = (I_\alpha * |v|^p) v^{p-1}.$$

Moreover, as $\mu \to \infty$, there holds

$$\|v_\mu\|_2^2 = \begin{cases} \mu^{-\frac{1}{p-1}} \left(\frac{N+\alpha-p(N-2)}{2p} S_p^{\frac{p}{p-1}} + O(\mu^{-\frac{q-2}{2(p-1)}}) \right), & \text{if} \quad q < \frac{2(2p+\alpha)}{2+\alpha}, \\ \mu^{-\frac{1}{p-1}} \left(\frac{N+\alpha-p(N-2)}{2p} S_p^{\frac{p}{p-1}} - \Theta(\mu^{-\frac{q-2}{2(p-1)}}) \right), & \text{if} \quad q \geq \frac{2(2p+\alpha)}{2+\alpha}, \end{cases}$$

$$\|\nabla v_\mu\|_2^2 = \mu^{-\frac{1}{p-1}} \left(\frac{N(p-1)-\alpha}{2p} S_p^{\frac{p}{p-1}} + O(\mu^{-\frac{q-2}{2(p-1)}}) \right),$$

and the least energy m_μ of the ground state satisfies

$$\frac{p-1}{2p} S_p^{\frac{p}{p-1}} - \mu^{\frac{1}{p-1}} m_\mu \sim \mu^{-\frac{q-2}{2(p-1)}},$$

as $\mu \to \infty$, where S_p is given in (2.12).

Our main results may be applied in other contexts. As an example, we consider

$$-\varepsilon^2 \Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u + |u|^{q-2}u, \text{ in } \mathbb{R}^N, \qquad (Q_\varepsilon)$$

where p, q and α are the same as before, and $\varepsilon > 0$ is a parameter. Set $v(x) = u(\varepsilon x)$, then we have

$$-\Delta v + v = \varepsilon^\alpha (I_\alpha * |v|^p)|v|^{p-2}v + |v|^{q-2}v, \text{ in } \mathbb{R}^N,$$

then as a direct consequence of main results in this paper, we have the following result.

Theorem 6.3. *If $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and $q \in (2, 2^*)$. Then the problem (Q_ε) admits a positive ground state $u_\varepsilon \in H^1(\mathbb{R}^N)$, which is radially symmetric and radially nonincreasing, and the rescaled family $u_\varepsilon(\varepsilon\cdot)$ converges in $H^1(\mathbb{R}^N)$ to the unique positive solution $v_0 \in H^1(\mathbb{R}^N)$ of the equation*

$$-\Delta v + v = v^{q-1}.$$

Moreover, as $\varepsilon \to 0$, there holds

$$\|u_\varepsilon\|_2^2 = \frac{2N - q(N-2)}{2q} S_q^{\frac{q}{q-2}} \varepsilon^N + O(\varepsilon^{N+\alpha}), \quad \text{if } q > \frac{2(2p+\alpha)}{2+\alpha},$$
$$\|u_\varepsilon\|_2^2 = \frac{2N - q(N-2)}{2q} S_q^{\frac{q}{q-2}} \varepsilon^N - \Theta(\varepsilon^{N+\alpha}), \quad \text{if } q \le \frac{2(2p+\alpha)}{2+\alpha},$$
$$\|\nabla u_\varepsilon\|_2^2 = \frac{N(q-2)}{2q} S_q^{\frac{q}{q-2}} \varepsilon^{N-2} + O(\varepsilon^{N-2+\alpha}),$$

and the least energy m_ε of the ground state satisfies

$$\frac{q-2}{2q} S_q^{\frac{q}{q-2}} - \varepsilon^{-N} m_\varepsilon \sim \varepsilon^\alpha,$$

as $\varepsilon \to 0$, where S_q is given in (2.14).

Data availability

No data was used for the research described in the article.

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Appendix A

In this appendix, we prove the following elementary lemma.

Lemma A.1. *Let $\eta > 0$ be a constant and $M(\varepsilon)$ is of class C^1 on $(0, \infty)$. Then the following statements hold true:*

(1) If $M(\varepsilon) \sim \varepsilon^\eta$ as $\varepsilon \to 0$, then there is $\varepsilon_0 > 0$ such that $M'(\varepsilon) > 0$ for $\varepsilon \in (0, \varepsilon_0)$.
(2) If $M(\varepsilon) \sim \varepsilon^{-\eta}$ as $\varepsilon \to 0$, then there is $\varepsilon_0 > 0$ such that $M'(\varepsilon) < 0$ for $\varepsilon \in (0, \varepsilon_0)$.
(3) If $M(\varepsilon) \sim \varepsilon^{-\eta}$ as $\varepsilon \to \infty$, then there is $\varepsilon_\infty > 0$ such that $M'(\varepsilon) < 0$ for $\varepsilon \in (\varepsilon_\infty, \infty)$.
(4) If $M(\varepsilon) \sim \varepsilon^\eta$ as $\varepsilon \to \infty$, then there is $\varepsilon_\infty > 0$ such that $M'(\varepsilon) > 0$ for $\varepsilon \in (\varepsilon_\infty, \infty)$.

Proof. Since $\eta > 0$, we can choose $\delta > 0$ such that

$$\max\{1, \eta\} < \delta < \eta + 1.$$

To prove (1), we let $\tilde{M}(\varepsilon) = \varepsilon^{1-\delta} M(\varepsilon)$ for $\varepsilon \in (0, \infty)$ and $\tilde{M}(0) = 0$. Since

$$0 \le \tilde{M}(\varepsilon) \le C\varepsilon^{1-\delta+\eta} \to 0, \quad \text{as } \varepsilon \to 0,$$

it follows that $\tilde{M}(\varepsilon)$ is of class C^1 in $(0, \infty)$ and continuous on $[0, \infty)$. Moreover,

$$\tilde{M}'(0) := \lim_{\varepsilon \to 0^+} \frac{\tilde{M}(\varepsilon) - \tilde{M}(0)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{M(\varepsilon)}{\varepsilon^\delta} \ge \lim_{\varepsilon \to 0^+} c\varepsilon^{\eta-\delta} = +\infty.$$

Therefore, there exists $\varepsilon_0 > 0$ such that $\tilde{M}'(\varepsilon) > 0$ for any $\varepsilon \in (0, \varepsilon_0)$. That is,

$$\tilde{M}'(\varepsilon) = (1-\delta)\varepsilon^{-\delta} M(\varepsilon) + \varepsilon^{1-\delta} M'(\varepsilon) > 0,$$

which together with $\delta > 1$ implies that

$$M'(\varepsilon) > (\delta - 1)\varepsilon^{-1} M(\varepsilon) > 0.$$

The proof of (1) is complete.

To prove (2), we let $\tilde{M}(\varepsilon) = -\frac{\varepsilon^{1-\delta}}{M(\varepsilon)}$ for $\varepsilon \in (0, \infty)$ and $\tilde{M}(0) = 0$. Since

$$0 \ge \tilde{M}(\varepsilon) \ge -\frac{\varepsilon^{1-\delta}}{c\varepsilon^{-\eta}} \to 0, \quad \text{as } \varepsilon \to 0,$$

it follows that $\tilde{M}(\varepsilon)$ is of class C^1 in $(0, \infty)$ and continuous on $[0, \infty)$. Moreover,

$$\tilde{M}'(0) := \lim_{\varepsilon\to 0^+} \frac{\tilde{M}(\varepsilon) - \tilde{M}(0)}{\varepsilon} = -\lim_{\varepsilon\to 0^+} \frac{\varepsilon^{-\delta}}{M(\varepsilon)} \le -\lim_{\varepsilon\to 0^+} \frac{\varepsilon^{-\delta}}{C\varepsilon^{-\eta}} = -\infty.$$

Therefore, there exists $\varepsilon_0 > 0$ such that $\tilde{M}'(\varepsilon) < 0$ for any $\varepsilon \in (0, \varepsilon_0)$. That is,

$$\tilde{M}'(\varepsilon) = -\frac{(1-\delta)\varepsilon^{-\delta} M(\varepsilon) - \varepsilon^{1-\delta} M'(\varepsilon)}{M(\varepsilon)^2} < 0,$$

which together with $\delta > 1$ implies that

$$M'(\varepsilon) < (1-\delta)\varepsilon^{-1} M(\varepsilon) < 0.$$

The proof of (2) is complete.

To prove (3), we let $\tilde{M}(\varepsilon) = -\varepsilon^{1-\delta} M(\frac{1}{\varepsilon})$ for $\varepsilon \in (0, \infty)$ and $\tilde{M}(0) = 0$. Since

$$0 \ge \tilde{M}(\varepsilon) \ge -C\varepsilon^{1-\delta+\eta} \to 0, \quad \text{as } \varepsilon \to 0,$$

it follows that $\tilde{M}(\varepsilon)$ is of class C^1 in $(0, \infty)$ and continuous on $[0, \infty)$. Moreover,

$$\tilde{M}'(0) := \lim_{\varepsilon\to 0^+} \frac{\tilde{M}(\varepsilon) - \tilde{M}(0)}{\varepsilon} \le -c \lim_{\varepsilon\to 0^+} \varepsilon^{\mu-\delta} = -\infty.$$

Therefore, there exists $\varepsilon_0 > 0$ such that $\tilde{M}'(\varepsilon) < 0$ for any $\varepsilon \in (0, \varepsilon_0)$. That is,

$$\tilde{M}'(\varepsilon) = -(1-\delta)\varepsilon^{-\delta} M(\frac{1}{\varepsilon}) + \varepsilon^{-1-\delta} M'(\frac{1}{\varepsilon}) < 0,$$

which together with $\delta > 1$ implies that

$$M'(\frac{1}{\varepsilon}) < (1-\delta)\varepsilon M(\frac{1}{\varepsilon}) < 0.$$

Take $\varepsilon_\infty = \frac{1}{\varepsilon_0}$, then $M'(\varepsilon) < 0$ for $\varepsilon \in (\varepsilon_\infty, \infty)$. The proof of (3) is complete.

To prove (4), we let $\tilde{M}(\varepsilon) = \frac{\varepsilon^{1-\delta}}{M(\frac{1}{\varepsilon})}$ for $\varepsilon \in (0, \infty)$ and $\tilde{M}(0) = 0$. Since

$$0 \le \tilde{M}(\varepsilon) \le \frac{\varepsilon^{1-\delta}}{c\varepsilon^{-\eta}} \to 0, \quad \text{as } \varepsilon \to 0,$$

it follows that $\tilde{M}(\varepsilon)$ is of class C^1 in $(0, \infty)$ and continuous on $[0, \infty)$. Moreover,

$$\tilde{M}'(0) := \lim_{\varepsilon\to 0^+} \frac{\tilde{M}(\varepsilon) - \tilde{M}(0)}{\varepsilon} \ge \lim_{\varepsilon\to 0^+} \frac{\varepsilon^{-\delta}}{C\varepsilon^{-\eta}} = +\infty.$$

Therefore, there exists $\varepsilon_0 > 0$ such that $\tilde{M}'(\varepsilon) > 0$ for any $\varepsilon \in (0, \varepsilon_0)$. That is,

$$\tilde{M}'(\varepsilon) = \frac{(1-\delta)\varepsilon^{-\delta} M(\frac{1}{\varepsilon}) + \varepsilon^{-1-\delta} M'(\frac{1}{\varepsilon})}{M(\frac{1}{\varepsilon})^2} > 0,$$

which together with $\delta > 1$ implies that

$$M'(\frac{1}{\varepsilon}) > (\delta - 1)\varepsilon M(\frac{1}{\varepsilon}) > 0.$$

Take $\varepsilon_\infty = \frac{1}{\varepsilon_0}$, then $M'(\varepsilon) > 0$ for $\varepsilon \in (\varepsilon_\infty, \infty)$. The proof of (4) is complete. □

References

[1] T. Akahori, S. Ibrahim, N. Ikoma, H. Kikuchi, H. Nawa, Uniqueness and nondegeneracy of ground states to nonlinear scalar field equations involving the Sobolev critical exponent in their nonlinearities for high frequencies, Calc. Var. Partial Differ. Equ. 58 (2019) 120, 32 pp.
[2] T. Akahori, S. Ibrahim, H. Kikuchi, H. Nawa, Global dynamics above the ground state energy for the combined power type nonlinear Schrodinger equations with energy critical growth at low frequencies, Mem. Am. Math. Soc. 272 (1331) (2021), v+130 pp.
[3] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Ration. Mech. Anal. 82 (1983) 313–345.
[4] C.G. Böhmer, T. Harko, Can dark matter be a Bose–Einstein condensate?, J. Cosmol. Astropart. Phys. (June 2007) 025.
[5] H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Am. Math. Soc. 88 (1983) 486–490.
[6] D. Cassani, J. Van Schaftingen, Jianjun Zhang, Groundstates for Choquard type equations with Hardy-Littlewood-Sobolev lower critical exponent, Proc. R. Soc. Edinb., Sect. A 150 (2020) 1377–1400.
[7] T. Cazenave, P.-L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Commun. Math. Phys. 85 (1982) 549–561.
[8] P.-H. Chavanis, Mass-radius relation of Newtonian self-gravitating Bose-Einstein condensates with short-range interactions: I. Analytical results, Phys. Rev. D 84 (2011) 043531.
[9] P.-H. Chavanis, Self-gravitating Bose-Einstein condensates, in: Quantum Aspects of Black Holes, in: Fundam. Theor. Phys., vol. 178, Springer, Cham, 2015, pp. 151–194.
[10] M. Coles, S. Gustafson, Solitary waves and dynamics for subcritical perturbations of energy critical NLS, Publ. Res. Inst. Math. Sci. 56 (2020) 647–699.
[11] S. Dovetta, E. Serra, P. Tilli, Action versus energy ground states in nonlinear Schrödinger equations, Math. Ann. 385 (2023) 1545–1576.
[12] Du Lele, Minbo Yang, Uniqueness and nondegeneracy of solutions for a critical nonlocal equation, Discrete Contin. Dyn. Syst. 39 (10) (2019) 5847–5866.
[13] J. Duoandikoetxea, Fractional integrals on radial functions with applications to weighted inequalities, Ann. Mat. Pura Appl. (4) 192 (2013) 553–568.
[14] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1983, xiii+513 pp.
[15] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry. I, J. Funct. Anal. 74 (1987) 160–197.
[16] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry. II, J. Funct. Anal. 94 (1990) 308–348.
[17] Y. Il'yasov, On orbital stability of the physical ground states of the NLS equations, arXiv:2103.16353v2.
[18] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, Nonlinear Anal. 28 (1997) 1633–1659.
[19] L. Jeanjean, J. Jendrej, T. Le, N. Visciglia, Orbital stability of ground states for a Sobolev critical Schrödinger equation, J. Math. Pures Appl. (9) 164 (2022) 158–179.
[20] L. Jeanjean, T. Le, Multiple normalized solutions for a Sobolev critical Schrödinger equation, Math. Ann. 384 (2022) 101–134.
[21] L. Jeanjean, Sheng-Shen Lu, On global minimizers for a mass constrained problem, Calc. Var. Partial Differ. Equ. 61 (2022) 214, 18 pp.
[22] L. Jeanjean, J. Zhang, X. Zhong, A global branch approach to normalized solutions for the Schrödinger equation, J. Math. Pures Appl. 183 (2024) 44–75.

[23] M. Lewin, S. Rota Nodari, The double-power nonlinear Schrödinger equation and its generalizations: uniqueness, non-degeneracy and applications, Calc. Var. Partial Differ. Equ. 59 (2020) 197, 49 pp.
[24] Xinfu Li, Shiwang Ma, Guang Zhang, Existence and qualitative properties of solutions for Choquard equations with a local term, Nonlinear Anal., Real World Appl. 45 (2019) 1–25.
[25] Xinfu Li, Shiwang Ma, Choquard equations with critical nonlinearities, Commun. Contemp. Math. 22 (2019) 1950023.
[26] Xinfu Li, Existence and symmetry of normalized ground state to Choquard equation with local perturbation, arXiv: 2103.07026v1.
[27] Xinfu Li, Standing waves to upper critical Choquard equation with a local perturbation: multiplicity, qualitative properties and stability, Adv. Nonlinear Anal. 11 (2022) 1134–1164.
[28] Xinfu Li, Nonexistence, existence and symmetry of normalized ground states to Choquard equations with a local perturbation, Complex Var. Elliptic Equ. 68 (2023) 578–602.
[29] Xinfu Li, Jianguang Bao, Wenguang Tang, Normalized solutions to lower critical Choquard equation with a local perturbation, Discrete Contin. Dyn. Syst., Ser. B 28 (2023) 3216–3232.
[30] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1976/77) 93–105.
[31] E.H. Lieb, M. Loss, Analysis, American Mathematical Society, Providence, RI, 2001.
[32] P.H. Lions, The concentration-compactness principle in the calculus of variations: the locally compact cases, part I and part II, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 1 (1984) 223–283.
[33] X.Q. Liu, J.Q. Liu, Z.-Q. Wang, Quasilinear elliptic equations with critical growth via perturbation method, J. Differ. Equ. 254 (2013) 102–124.
[34] Zeng Liu, V. Moroz, Limit profiles for singularly perturbed Choquard equations with local repulsion, Calc. Var. Partial Differ. Equ. 61 (2022) 160, 59 pp.
[35] S. Ma, V. Moroz, Asymptotic profiles for a nonlinear Schrödinger equation with critical combined powers nonlinearity, Math. Z. 304 (2023) 13, 26 pp.
[36] V. Moroz, C.B. Muratov, Asymptotic properties of ground states of scalar field equations with a vanishing parameter, J. Eur. Math. Soc. 16 (2014) 1081–1109.
[37] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, J. Funct. Anal. 265 (2013) 153–184.
[38] V. Moroz, J. Van Schaftingen, A guide to the Choquard equation, J. Fixed Point Theory Appl. 19 (2017) 773–813.
[39] A. Paredes, D.N. Olivieri, H. Michinel, From optics to dark matter: a review on nonlinear Schrödinger–Poisson systems, Phys. D: Nonlinear Phenom. 403 (2020) 132301.
[40] R. Ruffini, S. Bonazzola, Systems of self-gravitating particles in general relativity and the concept of an equation of state, Phys. Rev. 187 (1969) 1767–1783.
[41] J. Shatah, W. Strauss, Instability of nonlinear bound states, Commun. Math. Phys. 100 (1985) 173–190.
[42] D. Siegel, E. Talvila, Pointwise growth estimates of the Riesz potential, Dyn. Contin. Discrete Impuls. Syst. 5 (1999) 185–194.
[43] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case, J. Funct. Anal. 279 (2020) 108610.
[44] N. Soave, Normalized ground state for the NLS equations with combined nonlinearities, J. Differ. Equ. 269 (2020) 6941–6987.
[45] T. Tao, M. Visan, X. Zhang, The nonlinear Schrödinger equation with combined power-type nonlinearities, Commun. Partial Differ. Equ. 32 (2007) 1281–1343.
[46] K. Tintarev, K-H. Fieseler, Concentration Compactness: Functional-Analytic Grounds and Applications, Imperial College Press, 57 Shelton Street, London, 2007.
[47] X.Z. Wang, Cold Bose stars: self-gravitating Bose-Einstein condensates, Phys. Rev. D 64 (2001) 124009.
[48] J. Wei, Y. Wu, Normalized solutions for Schrödinger equations with critical Sobolev exponent and mixed nonlinearities, J. Funct. Anal. 283 (2022) 109574, 46 pp.
[49] J. Wei, Y. Wu, On some nonlinear Schrödinger equations in $\mathbb{R}^N$, Proc. R. Soc. Edinb., Sect. A 153 (2023) 1503–1528.
[50] M.I. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations, SIAM J. Math. Anal. 16 (1985) 472–491.
[51] Shuai Yao, Juntao Sun, Tsung-fang Wu, Normalized solutions for the Schrödinger equation with combined Hartree type and power nonlinearities, arXiv:2102.10268v1.
[52] Shuai Yao, Haibo Chen, V.D. Rădulescu, Juntao Sun, Normalized solutions for lower critical Choquard equations with critical Sobolev perturbation, SIAM J. Math. Anal. 54 (2022) 3696–3723.