

GROUP SCHEMES AND MOTIVIC SPECTRA*

BY

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ABSTRACT

By a theorem of Mandell, May, Schwede and Shipley [21] the stable homotopy theory of classical S^1 -spectra is recovered from orthogonal spectra. In this paper general linear, special linear, symplectic, orthogonal and special orthogonal motivic spectra are introduced and studied. It is shown that stable homotopy theory of motivic spectra is recovered from each of these types of spectra. An application is given for the localization functor $C_*\mathcal{F}r : \mathrm{SH}_{\mathrm{nis}}(k) \rightarrow \mathrm{SH}_{\mathrm{nis}}(k)$ in the sense of [15] that converts Morel–Voevodsky stable motivic homotopy theory $\mathrm{SH}(k)$ into the equivalent local theory of framed bispectra [15].

1. Introduction

In the 1990’s several approaches to the stable homotopy theory of S^1 -spectra were suggested. In [21] several comparison theorems relating the different constructions were proven showing that all of the known approaches to highly structured ring and module spectra are essentially equivalent.

Mandell, May, Schwede and Shipley [21] proved that the stable homotopy theory of classical topological S^1 -spectra is recovered from orthogonal spectra. In [24] Østvær conjectured that the stable homotopy theory of motivic spectra can be recovered from motivic GL-spectra, in which the role of the orthogonal

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groups as in topology [21] is played by the general linear group schemes GL_n -s. In this paper this conjecture is solved in the affirmative.

We follow [21] to develop the formal theory of diagram motivic spectra in Section 2. The framework allows lots of flexibility so that the reader can construct further interesting examples. For our purposes we work with diagram motivic spectra coming from group schemes GL_n -s, SL_n -s, Sp_n -s, O_n -s and SO_n -s (see Section 3). These group schemes act on motivic spheres. We also refer to the associated motivic spectra as general linear, special linear, symplectic, orthogonal and special orthogonal motivic spectra or just GL-, SL-, Sp-, O-, SO-motivic spectra.

One of the tricky concepts in the stable homotopy theory of classical symmetric spectra is that of semistability. Semistable symmetric spectra are important for understanding the difference between stable equivalences and maps inducing π_* -isomorphisms, that is, isomorphisms of the classical stable homotopy groups (in contrast with most other categories of spectra, not all stable equivalences of symmetric spectra induce π_* -isomorphisms). The same concept of semistability occurs in the stable homotopy theory of motivic spectra. We show in Section 4 that every GL-, SL- or Sp-motivic spectrum is semistable regarded as a symmetric motivic spectrum. This fact is the motivic counterpart of the classical result in topology saying that every orthogonal S^1 -spectrum of topological spaces is semistable.

We then define in Section 5 stable model structures on the categories of diagram motivic spectra. The main result of the paper is proven in Section 6 which compares ordinary/symmetric motivic spectra with GL-, SL-, Sp-, O- and SO-motivic spectra respectively (cf. Mandell–May–Schwede–Shipley [21, 0.1]).

THEOREM (Comparison): *Let k be any field. The following natural adjunctions between categories of T - and T^2 -spectra are all Quillen equivalences with respect to the stable model structure:*

- (1) $Sp_T(k) \rightleftarrows Sp_T^{GL}(k)$;
- (2) $Sp_{T^2}(k) \rightleftarrows Sp_{T^2}^{SL}(k)$;
- (3) $Sp_{T^2}(k) \rightleftarrows Sp_{T^2}^{Sp}(k)$;
- (4) $Sp_{T^2}(k) \rightleftarrows Sp_{T^2}^{SO}(k) \rightleftarrows Sp_{T^2}^O(k)$ if $\text{char } k \neq 2$.

An application of the Comparison Theorem is given in Section 7 for the localizing functor

$$C_*\mathcal{F}r : SH_{\text{nis}}(k) \rightarrow SH_{\text{nis}}(k)$$

in the sense of [15]. Recall that a new approach to the classical Morel–Voevodsky stable homotopy theory $\mathrm{SH}(k)$ was suggested in [15] and is based on the functor $C_*\mathcal{F}r$. This approach has nothing to do with any kind of motivic equivalences and is briefly defined as follows. We start with the local stable homotopy category of sheaves of S^1 -spectra $\mathrm{SH}_{S^1}^{\mathrm{nis}}(k)$. Then stabilizing $\mathrm{SH}_{S^1}^{\mathrm{nis}}(k)$ with respect to the endofunctor $\mathbb{G}_m^{\wedge 1} \wedge -$, we arrive at the triangulated category of bispectra $\mathrm{SH}_{\mathrm{nis}}(k)$. We then apply an explicit localization functor

$$C_*\mathcal{F}r : \mathrm{SH}_{\mathrm{nis}}(k) \rightarrow \mathrm{SH}_{\mathrm{nis}}(k)$$

that first takes a bispectrum E to its naive projective cofibrant resolution E^c and then one sets in each bidegree $C_*\mathcal{F}r(E)_{i,j} := C_*\mathcal{F}r(E_{i,j}^c)$. The localization functor $C_*\mathcal{F}r$ is isomorphic to the big framed motives localization functor \mathcal{M}_{fr}^b of [14] (see [15] as well). Then $\mathrm{SH}^{\mathrm{new}}(k)$ is defined as the category of $C_*\mathcal{F}r$ -local objects in $\mathrm{SH}_{\mathrm{nis}}(k)$. By [15, Section 2] $\mathrm{SH}^{\mathrm{new}}(k)$ is canonically equivalent to Morel–Voevodsky’s $\mathrm{SH}(k)$.

Using the Comparison Theorem above, we define new functors $C_*\mathcal{F}r^{\mathcal{G},n}$ on $\mathrm{SH}_{\mathrm{nis}}(k)$ that depend on $n \geq 0$ and the choice of the family of groups $\mathcal{G} = \{\mathrm{GL}_k\}_{k \geq 0}, \{\mathrm{SL}_{2k}\}_{k \geq 0}, \{\mathrm{Sp}_{2k}\}_{k \geq 0}, \{\mathrm{O}_{2k}\}_{k \geq 0}, \{\mathrm{SO}_{2k}\}_{k \geq 0}$. In Theorem 7.3 we prove that $C_*\mathcal{F}r$ and $C_*\mathcal{F}r^{\mathcal{G},n}$ are naturally isomorphic. As a result, one can incorporate linear algebraic groups into the theory of motivic infinite loop spaces and framed motives developed in [14].

Throughout the paper we denote by S a Noetherian scheme of finite dimension. We write Sm/S for the category of smooth separated schemes of finite type over S . Sm/S comes equipped with the Nisnevich topology [23, p. 95]. We denote by $(Shv_{\bullet}(Sm/S), \wedge, pt_+)$ the closed symmetric monoidal category of pointed Nisnevich sheaves on Sm/S . The category of pointed motivic spaces \mathbb{M}_{\bullet} is, by definition, the category $\Delta^{\mathrm{op}}Shv_{\bullet}(Sm/k)$ of pointed simplicial Nisnevich sheaves. Unless otherwise specified, we shall always deal with the flasque local (respectively motivic) model structure on \mathbb{M}_{\bullet} in the sense of [19]. Both model structures are weakly finitely generated in the sense of [10].

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2. Diagram motivic spaces and diagram motivic spectra

We refer the reader to [7] for basic facts of enriched category theory. We mostly adhere to [21] in this section. Suppose \mathcal{C} is a small category enriched over the closed symmetric monoidal category of pointed motivic spaces \mathbb{M}_\bullet . Following [21], a **motivic \mathcal{C} -space** or just a **\mathcal{C} -space** is an enriched functor $X : \mathcal{C} \rightarrow \mathbb{M}_\bullet$. The category of motivic \mathcal{C} -spaces and \mathbb{M}_\bullet -natural transformations between them is denoted by $[\mathcal{C}, \mathbb{M}_\bullet]$. In the language of enriched category theory $[\mathcal{C}, \mathbb{M}_\bullet]$ is the category of enriched functors from the \mathbb{M}_\bullet -category \mathcal{C} to the \mathbb{M}_\bullet -category \mathbb{M}_\bullet . When \mathcal{C} is enriched over unbased motivic spaces, we implicitly adjoin a base object $*$; in other words, we then understand $\mathcal{C}(a, b)$ to mean the union of the unbased motivic space of maps from a to b in \mathcal{C} and a disjoint basepoint.

2.1. Definition: For an object $a \in \mathcal{C}$, define the **evaluation functor** $\text{Ev}_a : [\mathcal{C}, \mathbb{M}_\bullet] \rightarrow \mathbb{M}_\bullet$ by

$$\text{Ev}_a(X) = X(a).$$

We also define the **shift desuspension functor** $F_a : \mathbb{M}_\bullet \rightarrow [\mathcal{C}, \mathbb{M}_\bullet]$ by

$$F_a(A) = \mathcal{C}(a, -) \wedge A$$

with $\mathcal{C}(a, -)$ the enriched functor represented by a . Then F_a is left adjoint to Ev_a .

For any $X \in [\mathcal{C}, \mathbb{M}_\bullet]$ there is a canonical isomorphism

$$X \cong \int^{c \in \mathcal{C}} \mathcal{C}(c, -) \wedge X(c) = \int^{c \in \mathcal{C}} F_c(X(c)).$$

If \mathcal{C} is a symmetric monoidal \mathbb{M}_\bullet -category with monoidal product \diamond and monoidal unit u , then $[\mathcal{C}, \mathbb{M}_\bullet]$ is a closed symmetric monoidal \mathbb{M}_\bullet -category with monoidal product

$$X \wedge Y = \int^{(a,b) \in \mathcal{C} \otimes \mathcal{C}} \mathcal{C}(a \diamond b, -) \wedge X(a) \wedge Y(b).$$

The monoidal unit is $\mathcal{C}(u, -)$. Moreover,

$$\mathcal{C}(a, -) \wedge \mathcal{C}(b, -) \cong \mathcal{C}(a \diamond b, -).$$

It follows that

$$F_a(A) \wedge F_b(B) \cong F_{a \diamond b}(A \wedge B), \quad A, B \in \mathbb{M}_\bullet.$$

2.2. *Definition:* Suppose $(\mathcal{C}, \diamond, u)$ is a symmetric monoidal \mathbb{M}_\bullet -category and R is a ring object in $[\mathcal{C}, \mathbb{M}_\bullet]$ with unit λ and product φ . Following [21, 1.9], a **\mathcal{C} -spectrum over R** is a \mathcal{C} -space $X \in [\mathcal{C}, \mathbb{M}_\bullet]$ together with maps $\sigma : X(a) \wedge R(b) \rightarrow X(a \diamond b)$, natural in a and b , such that the composite

$$X(a) \cong X(a) \wedge S^0 \xrightarrow{\text{id} \wedge \lambda} X(a) \wedge R(u) \xrightarrow{\sigma} X(a \diamond u) \cong X(a),$$

where $S^0 := pt_+$ is the identity and the following diagram commutes:

$$\begin{array}{ccc} X(a) \wedge R(b) \wedge R(c) & \xrightarrow{\sigma \wedge \text{id}} & X(a \diamond b) \wedge R(c) \\ \text{id} \wedge \varphi \downarrow & & \downarrow \sigma \\ X(a) \wedge R(b \diamond c) & \xrightarrow{\sigma} & X(a \diamond b \diamond c). \end{array}$$

The category of \mathcal{C} -spectra over R is denoted by $[\mathcal{C}, \mathbb{M}_\bullet]_R$. It is tensored and cotensored over \mathbb{M}_\bullet .

The following lemma is straightforward.

2.3. **LEMMA:** *Suppose \mathcal{C} is a symmetric monoidal \mathbb{M}_\bullet -category and R is a ring object in $[\mathcal{C}, \mathbb{M}_\bullet]$. Then the categories of (right) R -modules and of \mathcal{C} -spectra over R are isomorphic.*

A theorem of Day [9] also implies the following

2.4. **LEMMA:** *Let \mathcal{C} be a symmetric monoidal \mathbb{M}_\bullet -category and R a commutative ring object in $[\mathcal{C}, \mathbb{M}_\bullet]$. Then the category of R -modules $[\mathcal{C}, \mathbb{M}_\bullet]_R$ has a smash product \wedge_R and internal Hom-functor $\underline{\text{Hom}}_R$ under which it is a closed symmetric monoidal category with unit R .*

Let \mathcal{C} be a symmetric monoidal \mathbb{M}_\bullet -category and R a (not necessarily commutative) ring object in $[\mathcal{C}, \mathbb{M}_\bullet]$. Mandell, May, Schwede and Shipley [21, Section 2] suggested another description of the category of \mathcal{C} -spaces over R . Namely, $[\mathcal{C}, \mathbb{M}_\bullet]_R$ can be identified with the category of \mathcal{C}_R -spaces, where

$$\mathcal{C}_R(a, b) := [\mathcal{C}, \mathbb{M}_\bullet]_R(\mathcal{C}(b, -) \wedge R, \mathcal{C}(a, -) \wedge R).$$

The right-hand side refers to the \mathbb{M}_\bullet -object in $[\mathcal{C}, \mathbb{M}_\bullet]_R$. Composition is inherited from composition in $[\mathcal{C}, \mathbb{M}_\bullet]_R$. Thus \mathcal{C}_R can be regarded as the full \mathbb{M}_\bullet -subcategory of $[\mathcal{C}, \mathbb{M}_\bullet]_R^{\text{OP}}$ whose objects are the free R -modules $\mathcal{C}(a, -) \wedge R$.

By construction,

$$\begin{aligned} \mathcal{C}_R(a, b) &= [\mathcal{C}, \mathbb{M}_\bullet]_R(\mathcal{C}(b, -) \wedge R, \mathcal{C}(a, -) \wedge R) \cong [\mathcal{C}, \mathbb{M}_\bullet](\mathcal{C}(b, -), \mathcal{C}(a, -) \wedge R) \\ &\cong (\mathcal{C}(a, -) \wedge R)(b) \cong \int^{(f,g) \in \mathcal{C} \otimes \mathcal{C}} \mathcal{C}(f \diamond g, b) \wedge \mathcal{C}(a, f) \wedge R(g). \end{aligned}$$

If R is commutative, then \mathcal{C}_R is symmetric monoidal with monoidal product \diamond_R on objects being defined as the monoidal product \diamond in \mathcal{C} . Its unit object is the unit object u of \mathcal{C} . The product $f \diamond_R f'$ of morphisms $f : \mathcal{C}(b, -) \wedge R \rightarrow \mathcal{C}(a, -) \wedge R$ and $f' : \mathcal{C}(b', -) \wedge R \rightarrow \mathcal{C}(a', -) \wedge R$ is

$$\begin{aligned} f \diamond_R f' : \mathcal{C}(b \diamond b', -) \wedge R &\cong (\mathcal{C}(b, -) \wedge R) \wedge_R (\mathcal{C}(b', -) \wedge R) \\ &\rightarrow (\mathcal{C}(a, -) \wedge R) \wedge_R (\mathcal{C}(a', -) \wedge R) \cong \mathcal{C}(a \diamond a', -) \wedge R. \end{aligned}$$

The proof of the following fact literally repeats that of [21, 2.2], which is purely categorical and is not restricted by topological categories only.

2.5. THEOREM (Mandell–May–Schwede–Shipley): *Let \mathcal{C} be a symmetric monoidal \mathbb{M}_\bullet -category and R a ring object in $[\mathcal{C}, \mathbb{M}_\bullet]$. Then the categories $[\mathcal{C}, \mathbb{M}_\bullet]_R$ of \mathcal{C} -spectra over R and $[\mathcal{C}_R, \mathbb{M}_\bullet]$ of motivic \mathcal{C}_R -spaces are isomorphic. If R is commutative, then the isomorphism $[\mathcal{C}, \mathbb{M}_\bullet]_R \cong [\mathcal{C}_R, \mathbb{M}_\bullet]$ is an isomorphism of symmetric monoidal categories.*

3. Motivic spectra associated with group schemes

After collecting basic facts for \mathcal{C} -spectra over a ring object R in $[\mathcal{C}, \mathbb{M}_\bullet]$, where \mathcal{C} is a symmetric monoidal \mathbb{M}_\bullet -category, in this section we give particular examples we shall work with in this paper. The framework we have fixed above allows a lot of flexibility and we invite the interested reader to construct further examples. A canonical choice for a ring object, which we denote by \mathcal{S} or by $\mathcal{S}_{\mathcal{C}}$ if we want to specify the choice of the diagram \mathbb{M}_\bullet -category \mathcal{C} , is the motivic sphere spectrum

$$\mathcal{S} = (S^0, T, T^2, \dots),$$

where T^n is the Nisnevich sheaf $\mathbb{A}_S^n / (\mathbb{A}_S^n - 0)$. Another natural choice is the motivic sphere T^2 -spectrum

$$\mathcal{S} = (S^0, T^2, T^4, \dots)$$

consisting of the even dimensional spheres T^{2n} . The latter spectrum is necessary below when working with, say, special linear or symplectic groups. From the

homotopy theory viewpoint, stable homotopy categories of motivic T - and T^2 -spectra are Quillen equivalent (see, e.g., [25, 3.2]). Where it is possible we follow the terminology and notation of [21] in order to be consistent with the classical topological examples.

We should stress that in all our examples below the category of diagrams \mathcal{C} is defined in terms of group schemes. Our first example is elementary, but most important for our analysis.

3.1. *Example* (Ordinary motivic T -spectra): Let \mathcal{N} be the (unbased) category of non-negative integers $\mathbb{Z}_{\geq 0}$, with only “identity morphisms motivic spaces” between them. Precisely,

$$\mathcal{N}(m, n) = \begin{cases} pt, & m = n, \\ \emptyset, & m \neq n. \end{cases}$$

The symmetric monoidal structure is given by addition $m + n$, with 0 as unit. An \mathcal{N} -space is a sequence of based motivic spaces. The canonical enriched functor $\mathcal{S} = \mathcal{S}_{\mathcal{N}}$ takes $n \in \mathbb{Z}_{n \geq 0}$ to T^n . It is a ring object of $[\mathcal{N}, \mathbb{M}_{\bullet}]$, but it is not commutative since permutations of motivic spheres T^n are not identity maps. This is a typical difficulty in defining the smash product in stable homotopy theory. A **motivic T -spectrum** is an \mathcal{N} -spectrum over \mathcal{S} . Let $\mathrm{Sp}_{\mathcal{N}}^{\mathcal{N}}(\mathcal{S})$ denote the category of \mathcal{N} -spectra over \mathcal{S} . Since T^n is the n -fold smash product of T , the category $\mathrm{Sp}_{\mathcal{N}}^{\mathcal{N}}(\mathcal{S})$ is isomorphic to the category of ordinary motivic T -spectra $\mathrm{Sp}_T(\mathcal{S})$.

The shift desuspension functors to \mathcal{N} -spectra are given by

$$(F_m A)_n = A \wedge T^{n-m}$$

(by definition, $T^{n-m} = *$ if $n < m$). The smash product of \mathcal{N} -spaces (not \mathcal{N} -spectra!) is given by

$$(X \wedge Y)_n = \bigvee_{i=0}^n X_i \wedge Y_{n-i}.$$

The category $\mathcal{N}_{\mathcal{S}}$ such that an \mathcal{N} -spectrum is an $\mathcal{N}_{\mathcal{S}}$ -space has morphism motivic spaces

$$\mathcal{N}_{\mathcal{S}}(m, n) = T^{n-m}.$$

The category of ordinary motivic T^2 -spectra $\mathrm{Sp}_{T^2}^{\mathcal{N}}(\mathcal{S})$ is defined in a similar fashion.

As we have noticed above, $\mathcal{S}_{\mathcal{N}}$ is not commutative, and hence the category of \mathcal{N} -spectra $\mathrm{Sp}_T^{\mathcal{N}}(S)$ does not have a smash product that makes it a closed symmetric monoidal category. In all other examples below the ring object $\mathcal{S}_{\mathcal{C}} \in [\mathcal{C}, \mathbb{M}_{\bullet}]$ is commutative, and therefore the category of \mathcal{C} -spectra over $\mathcal{S}_{\mathcal{C}}$ is closed symmetric monoidal. The first classical example is that for symmetric spectra (we refer the reader to [20] for further details).

3.2. *Example* (Symmetric motivic T -spectra): Let Σ be the (unbased) category of finite sets $\mathbf{m} = \{1, \dots, m\}$. By definition, $\mathbf{0} := \emptyset$. Its morphisms motivic spaces $\Sigma(\mathbf{m}, \mathbf{n})$ are given by symmetric groups canonically regarded as group S -schemes. Precisely,

$$\Sigma(\mathbf{m}, \mathbf{n}) = \begin{cases} \Sigma_m, & m = n, \\ \emptyset, & m \neq n. \end{cases}$$

Notice that the underlying category associated with Σ is $\bigsqcup_{n \geq 0} \Sigma_n$. The symmetric monoidal structure on Σ is given by concatenation of sets $\mathbf{m} \sqcup \mathbf{n}$ and block sum of permutations, with $\mathbf{0}$ as unit. Commutativity of the monoidal product is given by the shuffle permutation $\chi_{m,n} : \mathbf{m} \sqcup \mathbf{n} \xrightarrow{\cong} \mathbf{n} \sqcup \mathbf{m}$ from the symmetric group Σ_{m+n} . The category $[\Sigma, \mathbb{M}_{\bullet}]$ is isomorphic to the category of symmetric sequences of pointed motivic spaces, i.e., the category of non-negatively graded pointed motivic spaces with symmetric groups actions.

The canonical enriched functor $\mathcal{S} = \mathcal{S}_{\Sigma}$ takes \mathbf{n} to T^n (Σ_n permutes the n copies of T or, equivalently, the coordinates of $\mathbb{A}_{\mathbb{S}}^n / (\mathbb{A}_{\mathbb{S}}^n - 0)$). It is a commutative ring object of $[\Sigma, \mathbb{M}_{\bullet}]$. A **symmetric motivic T -spectrum** is a Σ -spectrum over \mathcal{S} . Note that there is a canonical \mathbb{M}_{\bullet} -functor $\iota : \mathcal{N} \rightarrow \Sigma$ mapping n to \mathbf{n} such that $\mathcal{S}_{\mathcal{N}} = \mathcal{S}_{\Sigma} \circ \iota$.

The shift desuspension functors to symmetric spectra are given by

$$(F_m A)(\mathbf{n}) = \Sigma_{n+} \wedge_{\Sigma_{n-m}} (A \wedge T^{n-m}).$$

In turn, the smash product of Σ -spaces is given by

$$(X \wedge Y)(\mathbf{n}) = \bigvee_{i=0}^n \Sigma_{n+} \wedge_{\Sigma_i \times \Sigma_{n-i}} X(\mathbf{i}) \wedge Y(\mathbf{n} - \mathbf{i}).$$

The category $\Sigma_{\mathcal{S}}$ such that a Σ -spectrum is a $\Sigma_{\mathcal{S}}$ -space (see Theorem 2.5) has morphism spaces

$$\Sigma_{\mathcal{S}}(\mathbf{m}, \mathbf{n}) = \Sigma_{n+} \wedge_{\Sigma_{n-m}} T^{n-m}.$$

We shall write $\mathrm{Sp}_T^\Sigma(S)$ to denote the category of symmetric motivic T -spectra. The category of symmetric motivic T^2 -spectra $\mathrm{Sp}_{T^2}^\Sigma(S)$ is defined in a similar fashion.

3.3. Example (GL-motivic T -spectra): Let GL be the (unbased) category whose objects are the non-negative integers $\mathbb{Z}_{\geq 0}$. Its morphisms motivic spaces $\mathrm{GL}(m, n)$ are given by the following group \mathcal{S} -schemes:

$$\mathrm{GL}(m, n) = \begin{cases} \mathrm{GL}_m, & m = n, \\ \emptyset, & m \neq n. \end{cases}$$

The symmetric monoidal structure on GL is given by addition of integers and standard concatenation $\mathrm{GL}_m \times \mathrm{GL}_n \rightarrow \mathrm{GL}_{m+n}$ by block matrices. Commutativity of the monoidal product is given by the shuffle permutation matrix $\chi_{m,n} \in \mathrm{GL}_{m+n}$. The canonical enriched functor $\mathcal{S} = \mathcal{S}_{\mathrm{GL}}$ takes n to T^n (GL_n acts on $T^n = \mathbb{A}_S^n / (\mathbb{A}_S^n - 0)$ in a canonical way). It is a commutative ring object of $[\mathrm{GL}, \mathbb{M}_\bullet]$ because each GL_n contains Σ_n as permutation matrices. A **GL-motivic T -spectrum** is a GL -spectrum over \mathcal{S} . Note that there is a canonical \mathbb{M}_\bullet -functor $\iota : \Sigma \rightarrow \mathrm{GL}$ mapping \mathbf{n} to n and mapping permutations to their permutation matrices such that $\mathcal{S}_\Sigma = \mathcal{S}_{\mathrm{GL}} \circ \iota$.

The shift desuspension functors to GL -spectra are given by the induced motivic spaces (we refer the reader to [16] for basic facts on equivariant homotopy theory)

$$(1) \quad (F_m A)(n) = (\mathrm{GL}_n)_+ \wedge_{\mathrm{GL}_{n-m}} (A \wedge T^{n-m}).$$

In turn, the smash product of GL -spaces is given by

$$(X \wedge Y)(n) = \bigvee_{i=0}^n (\mathrm{GL}_n)_+ \wedge_{\mathrm{GL}_i \times \mathrm{GL}_{n-i}} X(i) \wedge Y(n-i).$$

The category $\mathrm{GL}_{\mathcal{S}}$ such that a GL -spectrum is a $\mathrm{GL}_{\mathcal{S}}$ -space (see Theorem 2.5) has morphism spaces

$$\mathrm{GL}_{\mathcal{S}}(m, n) = (\mathrm{GL}_n)_+ \wedge_{\mathrm{GL}_{n-m}} T^{n-m}.$$

A typical example of a GL -spectrum is the algebraic cobordism T -spectrum MGL (this follows from [25, Section 4]). We shall write $\mathrm{Sp}_T^{\mathrm{GL}}(S)$ to denote the category of GL -motivic T -spectra.

3.4. *Example (SL-motivic T^2 -spectra)*: In contrast to general linear groups, special linear groups contain only even permutations as their permutation matrices. We can equally define the “SL-category” as in Example 3.3 whose objects are all non-negative integers. The problem with such a \mathbb{M}_\bullet -category of diagrams is that it is not symmetric monoidal (unless characteristic is 2), and hence a problem with defining corresponding ring objects. To fix the problem, we work with even non-negative integers $2\mathbb{Z}_{\geq 0}$. We define morphisms motivic spaces $\text{SL}(2m, 2n)$ by the following group S -schemes:

$$\text{SL}(2m, 2n) = \begin{cases} \text{SL}_{2m}, & m = n, \\ \emptyset, & m \neq n. \end{cases}$$

We use the embedding $i_n : \Sigma_n \hookrightarrow \text{SL}_{2n}$ taking $\sigma \in \Sigma_n$ to the permutation matrix associated with $\tilde{\sigma} \in \Sigma_{2n}$, where

$$\tilde{\sigma}(2i - 1) = 2\sigma(i) - 1 \quad \text{and} \quad \tilde{\sigma}(2i) = 2\sigma(i).$$

With these embeddings of symmetric groups into even-dimensional special linear groups the diagram category SL becomes a symmetric monoidal \mathbb{M}_\bullet -category. The symmetric monoidal structure on SL is given by addition of integers and standard concatenation $\text{SL}_{2m} \times \text{SL}_{2n} \rightarrow \text{SL}_{2m+2n}$ by block matrices. Commutativity of the monoidal product is given by the shuffle permutation matrix $\chi_{2m, 2n} = i_n(\chi_{m, n}) \in \text{SL}_{2m+2n}$. The canonical enriched functor $\mathcal{S} = \mathcal{S}_{\text{SL}}$ takes $2n$ to T^{2n} (SL_{2n} acts on $T^{2n} = \mathbb{A}_S^{2n}/(\mathbb{A}_S^{2n} - 0)$ in a canonical way). It is a commutative ring object of $[\text{SL}, \mathbb{M}_\bullet]$ because each SL_{2n} contains Σ_n as permutation matrices defined above. An **SL-motivic T^2 -spectrum** is an SL -spectrum over \mathcal{S} . Note that there is a canonical \mathbb{M}_\bullet -functor $\iota : \Sigma \rightarrow \text{SL}$ mapping \mathbf{n} to $2n$ and $\sigma \in \Sigma_n$ to $i_n(\sigma)$ such that the symmetric sphere T^2 -spectrum (S^0, T^2, T^4, \dots) equals $\mathcal{S}_{\text{SL}} \circ \iota$. If there is no likelihood of confusion we shall also denote the symmetric sphere T^2 -spectrum (S^0, T^2, T^4, \dots) by \mathcal{S}_Σ whenever we work with T^2 -spectra. Notice that this T^2 -spectrum \mathcal{S}_Σ is a commutative ring object of $[\Sigma, \mathbb{M}_\bullet]$ and the category of right modules over \mathcal{S}_Σ is isomorphic to the category of symmetric T^2 -spectra $\text{Sp}_{T^2}^\Sigma(S)$.

The shift desuspension functors to SL -spectra are given by the induced motivic spaces

$$(2) \quad (F_{2m}A)(2n) = (\text{SL}_{2n})_+ \wedge_{\text{SL}_{2n-2m}} (A \wedge T^{2n-2m}).$$

In turn, the smash product of SL-spaces is given by

$$(X \wedge Y)(2n) = \bigvee_{i=0}^n (\mathrm{SL}_{2n})_+ \wedge_{\mathrm{SL}_{2i} \times \mathrm{SL}_{2n-2i}} X(2i) \wedge Y(2n - 2i).$$

The category $\mathrm{SL}_{\mathcal{S}}$ such that an SL-spectrum is an $\mathrm{SL}_{\mathcal{S}}$ -space (see Theorem 2.5) has morphism spaces

$$\mathrm{SL}_{\mathcal{S}}(2m, 2n) = (\mathrm{SL}_{2n})_+ \wedge_{\mathrm{SL}_{2n-2m}} T^{2n-2m}.$$

A typical example of an SL-spectrum is the algebraic special linear cobordism T^2 -spectrum MSL in the sense of Panin–Walter [25, Section 4]. We shall write $\mathrm{Sp}_{T^2}^{\mathrm{SL}}(\mathcal{S})$ to denote the category of SL-motivic T^2 -spectra.

3.5. *Example* (Symplectic motivic T^2 -spectra): Following [25, Section 6] we write the standard symplectic form on the trivial vector bundle of rank $2n$ as

$$\omega_{2n} = \begin{bmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & 0 & & & 0 & 1 \\ & & & & -1 & 0 \end{bmatrix}$$

The canonical symplectic isometry $(\mathcal{O}_S, \omega_{2n}) \cong (\mathcal{O}_S, \omega_2)^{\oplus n}$ gives rise to a natural action of Σ_n . It permutes the n orthogonal direct summands, and hence one gets an embedding $i_n : \Sigma_n \hookrightarrow \mathrm{Sp}_{2n}$, which sends permutations to the same permutation matrices as in Example 3.4. Let Sp have objects $2\mathbb{Z}_{\geq 0}$ and let morphisms motivic spaces $\mathrm{Sp}(2m, 2n)$ be defined by the following group \mathcal{S} -schemes:

$$\mathrm{Sp}(2m, 2n) = \begin{cases} \mathrm{Sp}_{2m}, & m = n, \\ \emptyset, & m \neq n. \end{cases}$$

With embeddings of symmetric groups into symplectic groups above the diagram category Sp becomes a symmetric monoidal \mathbb{M}_{\bullet} -category. The symmetric monoidal structure on Sp is given by addition of integers and standard concatenation $\mathrm{Sp}_{2m} \times \mathrm{Sp}_{2n} \rightarrow \mathrm{Sp}_{2m+2n}$ by block matrices. Commutativity of the monoidal product is given by the shuffle permutation matrix $\chi_{2m, 2n} \in \mathrm{Sp}_{2m+2n}$. The canonical enriched functor $\mathcal{S} = \mathcal{S}_{\mathrm{Sp}}$ takes $2n$ to T^{2n} (Sp_{2n} acts on $T^{2n} = \mathbb{A}_S^{2n}/(\mathbb{A}_S^{2n} - 0)$ in a canonical way). It is a commutative ring object of $[\mathrm{Sp}, \mathbb{M}_{\bullet}]$ because each Sp_{2n} contains Σ_n as permutation matrices defined above. A **symplectic motivic T^2 -spectrum** is an Sp -spectrum over \mathcal{S} . Note

that there is a canonical \mathbb{M}_\bullet -functor $\iota : \Sigma \rightarrow \text{Sp}$ mapping \mathbf{n} to $2n$ and $\sigma \in \Sigma_n$ to $i_n(\sigma)$ such that the symmetric sphere T^2 -spectrum $\mathcal{S}_\Sigma = (S^0, T^2, T^4, \dots)$ equals $\mathcal{S}_{\text{Sp}} \circ \iota$.

The shift desuspension functors to symplectic spectra are given by the induced motivic spaces

$$(3) \quad (F_{2m}A)(2n) = (\text{Sp}_{2n})_+ \wedge_{\text{Sp}_{2n-2m}} (A \wedge T^{2n-2m}).$$

In turn, the smash product of Sp -spaces is given by

$$(X \wedge Y)(2n) = \bigvee_{i=0}^n (\text{Sp}_{2n})_+ \wedge_{\text{Sp}_{2i} \times \text{Sp}_{2n-2i}} X(2i) \wedge Y(2n - 2i).$$

The category $\text{Sp}_{\mathcal{S}}$ such that an Sp -spectrum is an $\text{Sp}_{\mathcal{S}}$ -space (see Theorem 2.5) has morphism spaces

$$\text{Sp}_{\mathcal{S}}(2m, 2n) = (\text{Sp}_{2n})_+ \wedge_{\text{Sp}_{2n-2m}} T^{2n-2m}.$$

A typical example of a symplectic spectrum is the algebraic symplectic cobordism T^2 -spectrum MSp in the sense of Panin–Walter [25, Section 6]. We shall write $\text{Sp}_{T^2}^{\text{Sp}}(\mathcal{S})$ to denote the category of symplectic motivic T^2 -spectra.

In the next two examples we suppose $\frac{1}{2} \in \mathcal{S}$ and follow the terminology and notation of [8]. Denote by q_{2m} the **standard split quadratic form**

$$q_{2m} = x_1x_2 + x_3x_4 + \dots + x_{2m-1}x_{2m}.$$

We define $\text{O}_{2m} := \text{O}(q_{2m})$ and $\text{SO}_{2m} := \text{SO}(q_{2m})$.

3.6. *Example* (Orthogonal motivic T^2 -spectra): Let O have objects $2\mathbb{Z}_{\geq 0}$ and let morphisms motivic spaces $\text{O}(2m, 2n)$ be defined by the following group \mathcal{S} -schemes:

$$\text{O}(2m, 2n) = \begin{cases} \text{O}_{2m}, & m = n, \\ \emptyset, & m \neq n. \end{cases}$$

The corresponding embeddings of symmetric groups into orthogonal groups are the same with those of Example 3.4. Then the diagram category O becomes a symmetric monoidal \mathbb{M}_\bullet -category. The symmetric monoidal structure on O is given by addition of integers and standard concatenation $\text{O}_{2m} \times \text{O}_{2n} \rightarrow \text{O}_{2m+2n}$ by block matrices. Commutativity of the monoidal product is given by the shuffle permutation matrix $\chi_{2m,2n} \in \text{O}_{2m+2n}$. The canonical enriched functor $\mathcal{S} = \mathcal{S}_{\text{O}}$ takes $2n$ to T^{2n} (O_{2n} acts on $T^{2n} = \mathbb{A}_{\mathcal{S}}^{2n} / (\mathbb{A}_{\mathcal{S}}^{2n} - 0)$ in a canonical way).

It is a commutative ring object of $[\mathbf{O}, \mathbb{M}_\bullet]$ because each \mathbf{O}_{2n} contains Σ_n as permutation matrices defined above. An **orthogonal motivic T^2 -spectrum** is an \mathbf{O} -spectrum over \mathcal{S} . Note that there is a canonical \mathbb{M}_\bullet -functor $\iota : \Sigma \rightarrow \mathbf{O}$ mapping \mathbf{n} to $2n$ and $\sigma \in \Sigma_n$ to $i_n(\sigma)$ such that the symmetric sphere T^2 -spectrum $\mathcal{S}_\Sigma = (S^0, T^2, T^4, \dots)$ equals $\mathcal{S}_\mathbf{O} \circ \iota$.

The shift desuspension functors to orthogonal spectra are given by the induced motivic spaces

$$(4) \quad (F_{2m}A)(2n) = (\mathbf{O}_{2n})_+ \wedge_{\mathbf{O}_{2n-2m}} (A \wedge T^{2n-2m}).$$

In turn, the smash product of \mathbf{O} -spaces is given by

$$(X \wedge Y)(2n) = \bigvee_{i=0}^n (\mathbf{O}_{2n})_+ \wedge_{\mathbf{O}_{2i} \times \mathbf{O}_{2n-2i}} X(2i) \wedge Y(2n-2i).$$

The category $\mathbf{O}_\mathcal{S}$ such that an \mathbf{O} -spectrum is an $\mathbf{O}_\mathcal{S}$ -space (see Theorem 2.5) has morphism spaces

$$\mathbf{O}_\mathcal{S}(2m, 2n) = (\mathbf{O}_{2n})_+ \wedge_{\mathbf{O}_{2n-2m}} T^{2n-2m}.$$

We shall write $\mathrm{Sp}_{T^2}^{\mathbf{O}}(S)$ to denote the category of orthogonal motivic T^2 -spectra.

3.7. Example (SO-motivic T^2 -spectra): The definition of this type of motivic T^2 -spectra literally repeats Example 3.6 if we replace \mathbf{O}_{2n} with \mathbf{SO}_{2n} in all relevant places. The shift desuspension functors to SO-spectra are given by the induced motivic spaces

$$(5) \quad (F_{2m}A)(2n) = (\mathbf{SO}_{2n})_+ \wedge_{\mathbf{SO}_{2n-2m}} (A \wedge T^{2n-2m}).$$

The category $\mathbf{SO}_\mathcal{S}$ such that an SO-spectrum is an $\mathbf{SO}_\mathcal{S}$ -space (see Theorem 2.5) has morphism spaces

$$\mathbf{SO}_\mathcal{S}(2m, 2n) = (\mathbf{SO}_{2n})_+ \wedge_{\mathbf{SO}_{2n-2m}} T^{2n-2m}.$$

We shall write $\mathrm{Sp}_{T^2}^{\mathbf{SO}}(S)$ to denote the category of SO-motivic T^2 -spectra.

4. Semistable motivic spectra

One of the tricky concepts in the stable homotopy theory of classical symmetric spectra is that of semistability. The same concept of semistability occurs in the stable homotopy theory of motivic symmetric T - or T^2 -spectra.

Namely, following Røndigs, Spitzweck and Østvær [26], a motivic symmetric T -spectrum (likewise T^2 -spectrum) E is said to be *semistable* if the natural map

$$\varphi(E) : T \wedge E \rightarrow sh(E)$$

is a stable weak equivalence of underlying (non-symmetric) motivic spectra. In level n it is defined as the composite map

$$T \wedge E_n \xrightarrow{\cong} E_n \wedge T \rightarrow E_{n+1} \xrightarrow{\chi_{n,1}} E_{1+n}$$

of the twist isomorphism, the n th structure map of the spectrum E and the cyclic permutation $\chi_{n,1} = (1, 2, \dots, n + 1)$.

Similarly to the classical symmetric S^1 -spectra (see, e.g., [28, I.3.16]) a motivic symmetric T - or T^2 -spectrum X is semistable if for every n and every even permutation $\sigma \in \Sigma_n$ the action of σ on X_n coincides with the identity in the pointed motivic unstable homotopy category [26, 3.2].

It follows from Examples 3.3–3.7 that every G -spectrum, where $G \in \{GL, SL, Sp\}$, is a symmetric T - or T^2 -spectrum. It follows from [27, 3.2] that every orthogonal S^1 -spectrum of topological spaces is semistable. The following theorem is a motivic counterpart of that fact.

4.1. THEOREM: *Let $G \in \{GL, SL, Sp\}$. Then every G -spectrum is semistable as a symmetric T - or T^2 -spectrum.*

Proof. GL-, SL- or Sp-motivic spectra have the property that the action of the symmetric group Σ_n on the motivic spaces of GL-, SL- or Sp-motivic spectra factors through the action of GL_n , SL_{2n} and Sp_{2n} respectively. Therefore, even permutations are \mathbb{A}^1 -homotopic to identity (see [11, Section 2]).

In more detail, this means that if E is a G -spectrum and $\sigma \in \Sigma_n$ is an even permutation, then there is an \mathbb{A}^1 -homotopy $E_n \rightarrow \underline{\text{Hom}}(\mathbb{A}^1, E_n)$ between the action of σ and the identity map. It follows that the action of σ on E_n coincides with the identity in the pointed motivic unstable homotopy category, and hence E is semistable by [26, 3.2]. ■

As a consequence of the preceding theorem, we get rid of the semistability phenomenon for GL-, SL- or Sp-motivic spectra. Typical examples of such motivic spectra are MGL, MSL and MS Sp . It will follow from Theorem 6.1 that symmetric motivic spectra are Quillen equivalent to GL-, SL- or Sp-motivic

spectra. Therefore we can make symmetric motivic spectra GL-, SL- or Sp-motivic spectra by extending the group action and then compute the latter spectra within GL-, SL- or Sp-motivic spectra for which the phenomenon of semistability is irrelevant.

5. Model structures for \mathcal{C} -spectra

Throughout this section \mathcal{C} is a small category of diagrams enriched over \mathbb{M}_\bullet . Recall that \mathbb{M}_\bullet is equipped with the flasque motivic model structure in the sense of [19]. This model structure is simplicial, monoidal, proper, cellular and weakly finitely generated in the sense of [10]. It follows from [23, 3.2.13] that the smash product preserves motivic weak equivalences. Furthermore, \mathbb{M}_\bullet satisfies the monoid axiom in the sense of [29]. In the flasque model structure every sheaf of the form X/U is cofibrant, where $U \hookrightarrow X$ is a monomorphism in Sm/S . In particular, the sheaf T^n , $n \geq 0$, is flasque cofibrant.

Following [10, Section 4] $[\mathcal{C}, \mathbb{M}_\bullet]$ is equipped with the pointwise model structure, where a map f in $[\mathcal{C}, \mathbb{M}_\bullet]$ is a **pointwise motivic weak equivalence** (respectively a **pointwise fibration**) if $f(c)$ is a motivic weak equivalence (respectively fibration) in \mathbb{M}_\bullet for all $c \in \text{Ob}\mathcal{C}$. **Cofibrations** are defined as maps satisfying the left lifting property with respect to all pointwise acyclic fibrations.

5.1. PROPOSITION: *The following statements are true:*

- (1) $[\mathcal{C}, \mathbb{M}_\bullet]$ together with pointwise fibrations, pointwise motivic equivalences and cofibrations defined above is a simplicial cellular weakly finitely generated \mathbb{M}_\bullet -model category.
- (2) The pointwise model structure on $[\mathcal{C}, \mathbb{M}_\bullet]$ is proper.
- (3) If \mathcal{C} is a symmetric monoidal \mathbb{M}_\bullet -category, then $[\mathcal{C}, \mathbb{M}_\bullet]$ is a monoidal \mathbb{M}_\bullet -model category, and the monoid axiom in the sense of [29] holds.

Proof. (1). This follows from [10, 4.2, 4.4].

(2). Since \mathbb{M}_\bullet is right proper, then so is $[\mathcal{C}, \mathbb{M}_\bullet]$ by [10, 4.8]. Furthermore, \mathbb{M}_\bullet is strongly left proper in the sense of [10, 4.6]. By [10, 4.8] $[\mathcal{C}, \mathbb{M}_\bullet]$ is also left proper.

(3). This follows from [10, 4.4]. ■

5.2. COROLLARY: *Let \mathcal{C} be contained in a bigger \mathbb{M}_\bullet -category of diagrams \mathcal{D} . Then the canonical adjunction*

$$L : [\mathcal{C}, \mathbb{M}_\bullet] \rightleftarrows [\mathcal{D}, \mathbb{M}_\bullet] : U,$$

where L is the enriched left Kan extension and U is the forgetful functor, is a Quillen pair with respect to the pointwise model structure.

5.3. COROLLARY: *The categories of motivic T - and T^2 -spectra $\mathrm{Sp}_T^{\mathcal{N}}(S)$, $\mathrm{Sp}_{T^2}^{\mathcal{N}}(S)$, $\mathrm{Sp}_T^{\Sigma}(S)$, $\mathrm{Sp}_{T^2}^{\Sigma}(S)$, $\mathrm{Sp}_T^{\mathrm{GL}}(S)$, $\mathrm{Sp}_{T^2}^{\mathrm{SL}}(S)$, $\mathrm{Sp}_{T^2}^{\mathrm{O}}(S)$ and $\mathrm{Sp}_{T^2}^{\mathrm{SO}}(S)$ of Examples 3.1–3.7 are cellular weakly finitely generated proper \mathbb{M}_\bullet -model categories. Moreover, $\mathrm{Sp}_T^{\Sigma}(S)$, $\mathrm{Sp}_{T^2}^{\Sigma}(S)$, $\mathrm{Sp}_T^{\mathrm{GL}}(S)$, $\mathrm{Sp}_{T^2}^{\mathrm{SL}}(S)$, $\mathrm{Sp}_{T^2}^{\mathrm{O}}(S)$ and $\mathrm{Sp}_{T^2}^{\mathrm{SO}}(S)$ are monoidal \mathbb{M}_\bullet -model categories, and the monoid axiom holds for them.*

Proof. This follows from Proposition 5.1 and Theorem 2.5. ■

Recall that ordinary and symmetric motivic spectra have Quillen equivalent stable model structures (see, e.g., [20, 4.31]). We want to extend the stable model structure further to diagram spectra of Examples 3.3–3.7. To define it, we fix a symmetric monoidal diagram \mathbb{M}_\bullet -category \mathcal{C} together with a faithful strong symmetric monoidal functor of \mathbb{M}_\bullet -categories $\iota : \Sigma \rightarrow \mathcal{C}$ and a sphere ring spectrum $\mathcal{S} = \mathcal{S}_{\mathcal{C}}$ such that $\mathcal{S}_\Sigma = \mathcal{S}_{\mathcal{C}} \circ \iota$. We shall always assume that $\mathcal{S} = (S^0, K, K^{\wedge 2}, \dots)$ with $K = T$ or $K = T^2$. By Theorem 2.5 we identify the corresponding categories of spectra with categories $[\Sigma_{\mathcal{S}}, \mathbb{M}_\bullet]$ and $[\mathcal{C}_{\mathcal{S}}, \mathbb{M}_\bullet]$. As above, one has a natural adjunction

$$L : [\Sigma_{\mathcal{S}}, \mathbb{M}_\bullet] \rightleftarrows [\mathcal{C}_{\mathcal{S}}, \mathbb{M}_\bullet] : U.$$

5.4. *Definition:* Following Hovey [18, 8.7], define the **stable model structure** on $[\mathcal{C}_{\mathcal{S}}, \mathbb{M}_\bullet]$ to be the Bousfield localization with respect to \mathcal{P} of the pointwise model structure on $[\mathcal{C}_{\mathcal{S}}, \mathbb{M}_\bullet]$, where

$$\mathcal{P} = \{\lambda_n : F_{n+1}(C \wedge K) \rightarrow F_n C\}$$

as C runs through the domains and codomains of the generating cofibrations of \mathbb{M}_\bullet , and $n \geq 0$. The weak equivalences of the model category $[\mathcal{C}_{\mathcal{S}}, \mathbb{M}_\bullet]$ will be called **stable weak equivalences**. Note that if $\mathcal{C} = \Sigma$ then the stable model structure is nothing but the (flasque) stable model structure of symmetric spectra.

The preceding definition together with Corollary 5.2 and [18, 2.2] imply the following

5.5. PROPOSITION: *The canonical adjunction*

$$L : [\Sigma_S, \mathbb{M}_\bullet] \rightleftarrows [\mathcal{C}_S, \mathbb{M}_\bullet] : U,$$

where L is the enriched left Kan extension and U is the forgetful functor, is a Quillen pair with respect to the stable model structure.

Since ordinary T - or T^2 -spectra are Quillen equivalent to symmetric spectra (see [20, 4.31]), the preceding proposition implies the following

5.6. COROLLARY: *The canonical adjunction*

$$L : [\mathcal{N}_S, \mathbb{M}_\bullet] \rightleftarrows [\mathcal{C}_S, \mathbb{M}_\bullet] : U,$$

where L is the enriched left Kan extension and U is the forgetful functor, is a Quillen pair with respect to the stable model structure.

The main goal of the paper is to show that the adjunction of the previous proposition is a Quillen equivalence for \mathcal{C} being GL, SL, Sp, O and SO if we make a further assumption that the base scheme S is the spectrum $\text{Spec } k$ of a field k . This is treated in the next section.

6. The comparison theorem

Throughout this section k is any field. We shall freely operate with various equivalent models for $\text{SH}(k)$ like T -/ \mathbb{P}^1 -spectra or $(S^1, \mathbb{G}_m^{\wedge 1})$ -bispectra. It will always be clear which of the models is used.

The natural Quillen equivalences

$$\text{Sp}_T^{\mathcal{N}}(k) \rightleftarrows \text{Sp}_T^{\Sigma}(k), \quad \text{Sp}_{T^2}^{\mathcal{N}}(k) \rightleftarrows \text{Sp}_{T^2}^{\Sigma}(k)$$

between ordinary and symmetric motivic T - or T^2 -spectra are well-known (see, e.g., [20, 4.31]). The purpose of this section is to establish Quillen equivalences between spectra having a further structure given by various families of group schemes. Namely, we are now in a position to formulate the main result of the paper which compares ordinary/symmetric motivic spectra with GL-, SL-, Sp-, O- and SO-motivic spectra respectively (cf. Mandell–May–Schwede–Shipley [21, 0.1]).

6.1. THEOREM (Comparison): *The following natural adjunctions between categories of T - and T^2 -spectra are all Quillen equivalences with respect to the stable model structure of Definition 5.4:*

- (1) $\mathrm{Sp}_T^{\mathcal{N}}(k) \rightleftarrows \mathrm{Sp}_T^{\mathrm{GL}}(k);$
- (2) $\mathrm{Sp}_{T^2}^{\mathcal{N}}(k) \rightleftarrows \mathrm{Sp}_{T^2}^{\mathrm{SL}}(k);$
- (3) $\mathrm{Sp}_{T^2}^{\mathcal{N}}(k) \rightleftarrows \mathrm{Sp}_{T^2}^{\mathrm{SP}}(k);$
- (4) $\mathrm{Sp}_{T^2}^{\mathcal{N}}(k) \rightleftarrows \mathrm{Sp}_{T^2}^{\mathrm{SO}}(k) \rightleftarrows \mathrm{Sp}_{T^2}^{\mathrm{O}}(k)$ if $\mathrm{char} k \neq 2.$

We postpone its proof but first verify several statements which are of independent interest. Recall that a motivic space $A \in \mathbb{M}_\bullet$ is an \mathbb{A}^1 - n -**connected** if the Nisnevich sheaves $\pi_i^{\mathbb{A}^1}(A) \cong *$ for $i \leq n$. For any $B \in \mathrm{SH}(k)$, denote by $\pi_{i,n}^{\mathbb{A}^1}(B)$ the sheaf associated to the presheaf

$$U \in \mathrm{Sm}/S \mapsto \mathrm{SH}(k)(U_+ \wedge S^{i-n} \wedge \mathbb{G}_m^{\wedge n}, B).$$

B is said to be **connected** if $\pi_{i,n}^{\mathbb{A}^1}(B) = 0$ for $i < n$. We also set

$$\mathrm{SH}(k)_{\geq \ell} := \Sigma_{S^1}^\ell \mathrm{SH}(k)_{\geq 0}$$

and refer to the objects of $\mathrm{SH}(k)_{\geq \ell}$ as $(\ell - 1)$ -**connected**. We define the category of $(\ell - 1)$ -connected S^1 -spectra $\mathrm{SH}_{S^1}(k)_{\geq \ell}$ in a similar fashion. We say that a motivic space $A \in \mathbb{M}_\bullet$ is **stably $(\ell - 1)$ -connected**, $\ell \geq 0$, if its suspension S^1 -spectrum is in $\mathrm{SH}_{S^1}(k)_{\geq \ell}$ (i.e., all its negative sheaves of stable homotopy groups are zero below ℓ). Finally, a motivic space $A \in \mathbb{M}_\bullet$ is $(\ell - 1)$ -**biconnected**, $\ell \geq 0$, if its suspension bispectrum (or its \mathbb{P}^1 -/ T -spectrum) is in $\mathrm{SH}(k)_{\geq \ell}$.

6.2. Remark: In the language of framed motives [14] if $A \in \mathbb{M}_\bullet$ is $(\ell - 1)$ -biconnected and the base field is (infinite) perfect, then the framed motive $M_{fr}(A^c)$ (respectively the motivic space $C_*Fr(A^c)^{\mathrm{gp}}$ with ‘gp’ standing for group completion of the sectionwise H -space $C_*Fr(A^c)$), where A^c is a cofibrant resolution of A in the projective model structure of spaces, is locally $(\ell - 1)$ -connected as an S^1 -spectrum (respectively as a motivic space).

It is well-known that the suspension bispectrum of a space is connected. The following statement is a further extension of this fact.¹

6.3. PROPOSITION: *Let $n > 0$ and let $A \in \mathbb{M}_\bullet$ be an \mathbb{A}^1 - $(n - 1)$ -connected or stably $(n - 1)$ -connected pointed motivic space. Then A is $(n - 1)$ -biconnected.*

¹ The author thanks A. Ananyevskiy for pointing out a helpful argument used in the proof of this proposition.

Proof. Let A^f be a motivically fibrant replacement of A . First observe that the suspension S^1 -spectrum $\Sigma_{S^1}^\infty A^f$ is locally $(n - 1)$ -connected. Indeed, the zeroth space of the spectrum is locally $(n - 1)$ -connected by assumption, and hence each m th space $A \wedge S^m$ of the spectrum is locally $(m + n - 1)$ -connected. Morel’s stable \mathbb{A}^1 -connectivity theorem [22] implies $\Sigma_{S^1}^\infty(A^f)$ is motivically $(n - 1)$ -connected.

Since A^f is locally $(n - 1)$ -connected by assumption, it follows that each S^1 -spectrum $\Sigma_{S^1}^\infty(A \wedge \mathbb{G}_m^{\wedge q})$, $q \geq 0$, is locally $(n - 1)$ -connected. By Morel’s stable \mathbb{A}^1 -connectivity theorem [22] each $\Sigma_{S^1}^\infty(A^f \wedge \mathbb{G}_m^{\wedge q})$, $q \geq 0$, is motivically $(n - 1)$ -connected. Let $B = (B(0), B(1), \dots)$ denote a level motivically fibrant replacement of the bispectrum of $\Sigma_{\mathbb{G}_m^{\wedge 1}}^\infty \Sigma_{S^1}^\infty A^f$. Then each weight S^1 -spectrum $B(q)$ is motivically fibrant and locally $(n - 1)$ -connected.

Since B is a levelwise motivically fibrant bispectrum, then its stabilization in the $\mathbb{G}_m^{\wedge 1}$ -direction $\Theta_{\mathbb{G}_m^{\wedge 1}}^\infty B$ is motivically fibrant. We have that $\Theta_{\mathbb{G}_m^{\wedge 1}}^\infty B$ is a fibrant replacement of $\Sigma_{\mathbb{G}_m^{\wedge 1}}^\infty \Sigma_{S^1}^\infty A$. By definition, the q th weight S^1 -spectrum $\Theta_{\mathbb{G}_m^{\wedge 1}}^\infty B(q)$ is the colimit of the sequence

$$B(q) \rightarrow \Omega_{\mathbb{G}_m^{\wedge 1}} B(q + 1) \rightarrow \Omega_{\mathbb{G}_m^{\wedge 2}} B(q + 2) \rightarrow \dots$$

Since $B(q + i)$ is a motivically fibrant locally $(n - 1)$ -connected S^1 -spectrum, then so is $\Omega_{\mathbb{G}_m^{\wedge i}} B(q + i)$ by [13, A.2]. We see that each $\Theta_{\mathbb{G}_m^{\wedge 1}}^\infty B(q)$ is locally $(n - 1)$ -connected. Using [13, A.2] this is enough to conclude that $\Theta_{\mathbb{G}_m^{\wedge 1}}^\infty B \in \text{SH}(k)_{\geq n}$, and hence $\Sigma_{\mathbb{G}_m^{\wedge 1}}^\infty \Sigma_{S^1}^\infty A \in \text{SH}(k)_{\geq n}$.

The proof for stably $(n - 1)$ -connected motivic spaces is similar to that for \mathbb{A}^1 - $(n - 1)$ -connected spaces. ■

The proof of the preceding proposition also implies the following

6.4. COROLLARY: *Under the assumptions of Proposition 6.3 the space $A \wedge C$ is $(n - 1)$ -biconnected for any $C \in \mathbb{M}_\bullet$.*

The next result is crucial for proving Theorem 6.1.

6.5. THEOREM: *Given a pointed motivic space $C \in \mathbb{M}_\bullet$, the following natural maps are all stable motivic equivalences of ordinary motivic T - and T^2 -spectra:*

- (1) $\lambda_n : F_{n+1}(C \wedge T) \rightarrow F_n C$, where the shift desuspension functors are defined by (1) in Example 3.3 for GL -spectra;
- (2) $\lambda_n : F_{2n+2}(C \wedge T^2) \rightarrow F_{2n} C$, where the shift desuspension functors are defined by (2) in Example 3.4 for SL -spectra;

- (3) $\lambda_n : F_{2n+2}(C \wedge T^2) \rightarrow F_{2n}C$, where the shift desuspension functors are defined by (3) in Example 3.5 for symplectic spectra;
- (4) $\lambda_n : F_{2n+2}(C \wedge T^2) \rightarrow F_{2n}C$, where the shift desuspension functors are defined by (4) in Example 3.6 for orthogonal spectra provided that $\text{char } k \neq 2$;
- (5) $\lambda_n : F_{2n+2}(C \wedge T^2) \rightarrow F_{2n}C$, where the shift desuspension functors are defined by (5) in Example 3.7 for SO-spectra provided that $\text{char } k \neq 2$.

6.6. Remark: If G is a linear algebraic group over a field k , and H is a closed subgroup, then by G/H we mean the unpointed Nisnevich sheaf associated with the presheaf $U \mapsto G(U)/H(U)$. If G and H are smooth and all H -torsors are Zariski locally trivial, then the sheaf G/H is represented by a scheme (see [3, p. 275]). If there is no likelihood of confusion, we shall denote the scheme by the same symbol G/H . By [3, p. 275] this happens, for example, if $H = \text{GL}_n, \text{SL}_n$ or Sp_{2n} . In turn, if $\text{char } k \neq 2$ then it is proved similarly to [5, 3.1.9] that the torsors $\text{O}_{2n+2k} \rightarrow \text{O}_{2n+2k}/\text{O}_{2n}$ and $\text{SO}_{2n+2k} \rightarrow \text{SO}_{2n+2k}/\text{SO}_{2n}$, $k, n > 0$, are Zariski locally trivial. Note that the schemes $\text{GL}_{n+k}/\text{GL}_n, \text{SL}_{2n+2k}/\text{SL}_{2n}, \text{Sp}_{2n+2k}/\text{Sp}_{2n}, \text{O}_{2n+2k}/\text{O}_{2n}, \text{SO}_{2n+2k}/\text{SO}_{2n}$, $k, n > 0$, are all smooth (see [5, 2.3.1] and [6, p. 638]). For the latter two we assume $\text{char } k \neq 2$. As mentioned above, they all represent the corresponding quotient Nisnevich sheaves (see [5, 2.3.1] as well).

Proof of Theorem 6.5. (1). This is the case of GL -motivic spectra. By definition (see (1)),

$$(F_n C)(q) = (\text{GL}_q)_+ \wedge_{\text{GL}_{q-n}} (C \wedge T^{q-n}).$$

For $q \geq n + 1$, $\lambda_n(q)$ is the canonical quotient map

$$\begin{aligned} (\text{GL}_q)_+ \wedge_{\text{GL}_{q-n-1}} (C \wedge T \wedge T^{q-n-1}) &= (\text{GL}_q)_+ \wedge_{\text{GL}_{q-n-1}} (C \wedge T^{q-n}) \\ &\rightarrow (\text{GL}_q)_+ \wedge_{\text{GL}_{q-n}} (C \wedge T^{q-n}). \end{aligned}$$

Since $T^n \wedge -$ reflects stable motivic equivalences of ordinary T -spectra by [20, 3.18], our statement reduces to showing that $T^n \wedge \lambda_n$ is a stable motivic equivalence in $\text{Sp}_T^{\mathcal{N}}(k)$.

The map $T^n \wedge \lambda_n$ takes the form

$$(\text{GL}_q)_+ \wedge_{\text{GL}_{q-n-1}} (C \wedge T^q) \rightarrow (\text{GL}_q)_+ \wedge_{\text{GL}_{q-n}} (C \wedge T^q).$$

Since GL_q acts on T^q , it follows from [16, 1.2] that the latter map is isomorphic to the map

$$\lambda'_q : (\mathrm{GL}_q/\mathrm{GL}_{q-n-1})_+ \wedge C \wedge T^q \rightarrow (\mathrm{GL}_q/\mathrm{GL}_{q-n})_+ \wedge C \wedge T^q.$$

Here $\mathrm{GL}_q/\mathrm{GL}_{q-n-1}, \mathrm{GL}_q/\mathrm{GL}_{q-n}$ are smooth schemes of Remark 6.6. Set

$$F'_{n+1}(C \wedge T) := (*, \cdot, \cdot, *, (\mathrm{GL}_{n+1})_+ \wedge C \wedge T^{n+1}, (\mathrm{GL}_{n+2}/\mathrm{GL}_1)_+ \wedge C \wedge T^{n+2}, \\ (\mathrm{GL}_{n+3}/\mathrm{GL}_2)_+ \wedge C \wedge T^{n+3}, \dots)$$

and

$$F'_n(C) := (*, \cdot, \cdot, *, (\mathrm{GL}_n)_+ \wedge C \wedge T^n, (\mathrm{GL}_{n+1}/\mathrm{GL}_1)_+ \wedge C \wedge T^{n+1}, \\ (\mathrm{GL}_{n+2}/\mathrm{GL}_2)_+ \wedge C \wedge T^{n+2}, (\mathrm{GL}_{n+3}/\mathrm{GL}_3)_+ \wedge C \wedge T^{n+3}, \dots).$$

The structure of T -spectra on $F'_{n+1}(C \wedge T)$ and on $F'_n(C)$ are obvious. It is induced by the action of T on the right.

Consider a commutative diagram of ordinary motivic T -spectra

$$\begin{array}{ccc} sh^{-n-1}(\Sigma_T^\infty(C \wedge T \wedge T^n)) & \xrightarrow{\alpha} & F'_{n+1}(C \wedge T) \\ \downarrow & & \downarrow \lambda' \\ sh^{-n}(\Sigma_T^\infty(C \wedge T^n)) & \xrightarrow{\beta} & F'_n(C), \end{array}$$

where

$$sh^{-n}(\Sigma_T^\infty(C \wedge T^n)) = (*, \cdot, \cdot, *, C \wedge T^n, C \wedge T^{n+1}, \dots)$$

is the $(-n)$ th shift of $\Sigma_T^\infty C$, α, β are induced by the the following injective maps in \mathbb{M}_\bullet :

$$S^0 \rightarrow (\mathrm{GL}_q/\mathrm{GL}_{q-n-1})_+, \quad S^0 \rightarrow (\mathrm{GL}_q/\mathrm{GL}_{q-n})_+.$$

They send the basepoint of S^0 to $+$ and the unbasepoint to GL_{q-n-1} and GL_{q-n} respectively. Note that the left vertical arrow is a stable motivic equivalence in $\mathrm{Sp}_T^{\mathcal{N}}(k)$. Observe that α and β are isomorphic to counit andjunction maps

$$T^n \wedge F'_{n+1}(C \wedge T) \rightarrow T^n \wedge F_{n+1}^{GL}(C \wedge T), \quad T^n \wedge F'_n(C) \rightarrow T^n \wedge F_n^{GL}(C).$$

To show that $T^n \wedge \lambda_n$ is a stable motivic equivalence it is enough to show that α and β are stable motivic equivalences.

The map α fits in a level cofiber sequence of T -spectra

$$sh^{-n-1}(\Sigma_T^\infty(C \wedge T \wedge T^n)) \hookrightarrow F'_{n+1}(C \wedge T) \rightarrow F''_{n+1}(C \wedge T),$$

where $F''_{n+1}(C \wedge T)$ is the bispectrum

$$(*, \cdot, *, \text{GL}_{n+1} \wedge C \wedge T^{n+1}, \\ (\text{GL}_{n+2}/\text{GL}_1) \wedge C \wedge T^{n+2}, (\text{GL}_{n+3}/\text{GL}_2) \wedge C \wedge T^{n+3}, \dots),$$

where GL_{n+1} is pointed at the identity matrix and $\text{GL}_{n+i}/\text{GL}_{i-1}$ is pointed at GL_{i-1} .

We claim that $F''_{n+1}(C \wedge T)$ is isomorphic to zero in $\text{SH}(k)$. This is equivalent to saying that

$$F''(C \wedge T) := (\text{GL}_{n+1} \wedge C, (\text{GL}_{n+2}/\text{GL}_1) \wedge C \wedge T, (\text{GL}_{n+3}/\text{GL}_2) \wedge C \wedge T^2, \dots)$$

is isomorphic to zero in $\text{SH}(k)$ (we use here [20, 3.18]). Every T -spectrum $E = (E_0, E_1, \dots)$ has the layer filtration $E = \text{colim}_i L_i E$ with

$$L_n E = (E_0, \dots, E_{i-1}, E_i, E_i \wedge T, E_i \wedge T^2, \dots).$$

The i th layer $L_i F''(C \wedge T)$ of $F''(C \wedge T)$ is isomorphic to $\Sigma_T^\infty((\text{GL}_{n+1+i}/\text{GL}_i) \wedge C)$ in $\text{SH}(k)$.

By the proof of [4, 2.1.3] the “projection onto the first column map”

$$\text{GL}_n/\text{GL}_{n-1} \rightarrow \mathbb{A}^n \setminus 0$$

is a motivic equivalence of spaces. It follows from [4, 2.1.4] that $\mathbb{A}^n \setminus 0$ is $\mathbb{A}^{1-(n-2)}$ -connected for $n \geq 2$, and hence so is $\text{GL}_n/\text{GL}_{n-1}$. If we consider a fibre sequence of motivic spaces

$$\text{GL}_{n+l-1}/\text{GL}_{n-1} \rightarrow \text{GL}_{n+k+l-1}/\text{GL}_{n-1} \rightarrow \text{GL}_{n+k+l-1}/\text{GL}_{n+l-1}, \quad k, l \geq 0,$$

we conclude by induction that $\text{GL}_{n+k}/\text{GL}_{n-1}$ is $\mathbb{A}^{1-(n-2)}$ -connected as well for $n \geq 2$.

By Corollary 6.4, $\Sigma_T^\infty((\text{GL}_{n+i+1}/\text{GL}_i) \wedge C) \in \text{SH}(k)_{\geq i-1}$ if $i \geq 2$, and hence $L_i F''(C \wedge T) \in \text{SH}(k)_{\geq i-1}$. We see that $F''(C \wedge T) \in \bigcap_{i \in \mathbb{N}} \text{SH}(k)_{\geq i}$. This is only possible when $F''(C \wedge T) \cong 0$ in $\text{SH}(k)$, and our claim follows. Thus α is a stable equivalence, because its cofiber $F''_{n+1}(C \wedge T)$ is zero in $\text{SH}(k)$. Using the same arguments, β is a stable equivalence as well, and hence so is λ_n as stated.

The proof of (2), (3), (4), (5) literally repeats that of (1) if we use [6, 2.13] saying that $\text{SL}_n/\text{SL}_{n-1}$, $\text{Sp}_{2m}/\text{Sp}_{2m-2}$ (with $n = 2m$) are isomorphic to the odd-dimensional motivic sphere Q_{2n-1} which is, by definition, the affine quadric defined by the equation $\sum_{i=1}^n x_i y_i = 1$. By [2, p. 1892] Q_{2n-1} is \mathbb{A}^1 -equivalent

to $\mathbb{A}^n \setminus 0$, and hence it is $\mathbb{A}^{1-(n-2)}$ -connected for $n \geq 2$ by [4, 2.1.4]. Likewise, $\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$ (hence $\mathrm{O}_{2n+1}/\mathrm{O}_{2n}$ as well) is isomorphic to Q_{2n} by the proof of [6, 2.15], where Q_{2n} is the affine quadric defined by the equation

$$\sum_{i=1}^n x_i y_i = z(1-z).$$

Since Q_{2n} is \mathbb{A}^1 -equivalent to $S^n \wedge \mathbb{G}_m^{\wedge n}$ by [2, Theorem 2], it is $\mathbb{A}^{1-(n-1)}$ -connected for $n \geq 1$. Also, $\mathrm{SO}_{2n}/\mathrm{SO}_{2n-1}$ (hence $\mathrm{O}_{2n}/\mathrm{O}_{2n-1}$ as well) is isomorphic to Q_{2n-1} by [6, 2.13], and so it is $\mathbb{A}^{1-(n-2)}$ -connected for $n \geq 2$. ■

We shall need the following useful fact.

6.7. PROPOSITION: *Let $G \in \{\mathrm{GL}, \mathrm{SL}, \mathrm{Sp}, \mathrm{O}, \mathrm{SO}\}$. A map $f : X \rightarrow Y$ of G -spectra in the sense of Examples 3.3–3.7 is a stable equivalence in the sense of Definition 5.4 if and only if it is a stable motivic equivalence of ordinary motivic spectra.*

Proof. We prove the statement for GL -motivic T -spectra, because the proof for the other cases is similar. Denote by

$$\tilde{\mathcal{P}}^{\mathcal{N}} := \{\mathrm{cyl}(\lambda_n : F_{n+1}^{\mathcal{N}}(C \wedge T) \rightarrow F_n^{\mathcal{N}}T) \mid \lambda_n \in \mathcal{P}\},$$

where *cyl* refers to the ordinary mapping cylinder map, \mathcal{P} is the family of Definition 5.4 corresponding to ordinary T -spectra. Similarly, set

$$\tilde{\mathcal{P}}^{\mathrm{GL}} := \{\mathrm{cyl}(\lambda_n : F_{n+1}^{\mathrm{GL}}(C \wedge T) \rightarrow F_n^{\mathrm{GL}}T) \mid \lambda_n \in \mathcal{P}\},$$

where \mathcal{P} is the family of Definition 5.4 corresponding to GL -spectra. Then $\tilde{\mathcal{P}}^{\mathcal{N}}$ (respectively $\tilde{\mathcal{P}}^{\mathrm{GL}}$) is a family of cofibrations in $\mathrm{Sp}_T^{\mathcal{N}}(k)$ (respectively in $\mathrm{Sp}_T^{\mathrm{GL}}(k)$) with respect to the stable model structure. Also, the left Kan extension functor of Corollary 5.2

$$L : \mathrm{Sp}_T^{\mathcal{N}}(k) \rightarrow \mathrm{Sp}_T^{\mathrm{GL}}(k)$$

takes $\tilde{\mathcal{P}}^{\mathcal{N}}$ to $\tilde{\mathcal{P}}^{\mathrm{GL}}$.

The proof of Theorem 6.5 shows that the commutative square in $\mathrm{Sp}_T^{\mathcal{N}}(k)$

$$\begin{array}{ccc} F_{n+1}^{\mathcal{N}}(C \wedge T) & \xrightarrow{\lambda_n} & F_n^{\mathcal{N}}C \\ \alpha \downarrow & & \downarrow \beta \\ F_{n+1}^{\mathrm{GL}}(C \wedge T) & \xrightarrow{\lambda_n} & F_n^{\mathrm{GL}}C \end{array}$$

with vertical maps being the counit maps consists of stable motivic equivalences. Since the cylinder maps are preserved by the forgetful functor

$$U : \mathrm{Sp}_T^{\mathrm{GL}}(k) \rightarrow \mathrm{Sp}_T^{\mathcal{N}}(k),$$

it follows that $U(\widetilde{\mathcal{P}}^{\mathrm{GL}})$ is a family of injective stable motivic equivalences.

Let J be a family of generating trivial flasque cofibrations [19, 3.2(b)] for \mathbb{M}_\bullet . By [19, 3.10] domains and codomains of the maps in J are finitely presentable. Recall that the set of maps in $\mathrm{Sp}_T^{\mathcal{N}}(k)$ (respectively in $\mathrm{Sp}_T^{\mathrm{GL}}(k)$)

$$\mathcal{P}_J^{\mathcal{N}} := \bigcup_{n \geq 0} F_n^{\mathcal{N}}(J) \quad (\text{respectively } \mathcal{P}_J^{\mathrm{GL}} := \bigcup_{n \geq 0} F_n^{\mathrm{GL}}(J))$$

is a family of generating trivial cofibrations for the pointwise model structure of Proposition 5.1 (see, e.g., the proof of [10, 4.2]). By construction, $L(\mathcal{P}_J^{\mathcal{N}}) = \mathcal{P}_J^{\mathrm{GL}}$.

We set

$$\widehat{\mathcal{P}}^{\mathrm{GL}} := \{A \wedge \Delta[n]_+ \sqcup_{A \wedge \partial \Delta[n]_+} B \wedge \partial \Delta[n]_+ \rightarrow B \wedge \Delta[n]_+ \mid (A \rightarrow B) \in \widetilde{\mathcal{P}}^{\mathrm{GL}}, n \geq 0\}.$$

An augmented family of $\widetilde{\mathcal{P}}^{\mathrm{GL}}$ -horns is the following family of trivial cofibrations:

$$\Lambda(\mathcal{P}^{\mathrm{GL}}) = \mathcal{P}_J^{\mathrm{GL}} \cup \widehat{\mathcal{P}}^{\mathrm{GL}}.$$

Observe that domains and codomains of the maps in $\Lambda(\mathcal{P}^{\mathrm{GL}})$ are finitely presentable. It can be proven similarly to [18, 4.2] that a map $f : A \rightarrow B$ is a fibration in the stable model structure with fibrant codomain if and only if it has the right lifting property with respect to $\Lambda(\mathcal{P}^{\mathrm{GL}})$.

By [20, 2.12] a map $f : X \rightarrow Y$ in $\mathrm{Sp}_T^{\mathcal{N}}(k)$ is a stable motivic equivalence if and only if it induces a weak equivalence $f^* : \mathrm{Map}_*(Y, W) \rightarrow \mathrm{Map}_*(X, W)$ of Kan complexes for all stably fibrant injective T -spectra W . It follows that a pushout of an injective stable motivic equivalence is an injective stable motivic equivalence. Since all colimits in $\mathrm{Sp}_T^{\mathrm{GL}}(k)$ are computed in $\mathrm{Sp}_T^{\mathcal{N}}(k)$, it follows that a pushout of a coproduct of maps from $\Lambda(\mathcal{P}^{\mathrm{GL}})$ computed in $\mathrm{Sp}_T^{\mathrm{GL}}(k)$ is a stable motivic equivalence in $\mathrm{Sp}_T^{\mathcal{N}}(k)$, because every map of $\Lambda(\mathcal{P}^{\mathrm{GL}})$ is an injective stable motivic equivalence in $\mathrm{Sp}_T^{\mathcal{N}}(k)$. In particular, U sends $\Lambda(\mathcal{P}^{\mathrm{GL}})$ -cell complexes to stable motivic equivalence in $\mathrm{Sp}_T^{\mathcal{N}}(k)$.

We now apply the small object argument to the family $\Lambda(\mathcal{P}^{\text{GL}})$ in order to fit $f : X \rightarrow Y$ of the proposition into a commutative diagram

$$\begin{array}{ccc} U(X) & \longrightarrow & U(L_{\mathcal{P}}X) \\ U(f) \downarrow & & \downarrow U(L_{\mathcal{P}}f) \\ U(Y) & \longrightarrow & U(L_{\mathcal{P}}Y) \end{array}$$

with $X \rightarrow L_{\mathcal{P}}X$, $Y \rightarrow L_{\mathcal{P}}Y$ being $\Lambda(\mathcal{P}^{\text{GL}})$ -cell complexes and $L_{\mathcal{P}}X$, $L_{\mathcal{P}}Y$ stably fibrant GL-spectra (hence stably fibrant ordinary spectra by Corollary 5.6). Notice that $L_{\mathcal{P}}f$ is a level motivic equivalence. Our statement now follows. \blacksquare

Proof of Theorem 6.1. We only prove that

$$L : \text{Sp}_T^{\mathcal{N}}(k) \rightleftarrows \text{Sp}_T^{\text{GL}}(k) : U$$

is a Quillen equivalence with respect to the stable model structure, because the other cases are proved in a similar fashion.

The proof of Theorem 6.5 shows that the counit map $\beta_n : F_n^{\mathcal{N}}(C) \rightarrow U(F_n^{\text{GL}}(C))$ is a stable motivic equivalence in $\text{Sp}_T^{\mathcal{N}}(k)$ for any $C \in \mathbb{M}_{\bullet}$. Suppose $E \in \text{Sp}_T^{\mathcal{N}}(k)$ is cofibrant. Present it as

$$E = \text{colim}_n L_n^{\mathcal{N}}(E),$$

where each layer $L_n^{\mathcal{N}}(E) = (E_0, \dots, E_{n-1}, E_n, E_n \wedge T, \dots)$ is cofibrant as well. The canonical map $\varphi_n : F_n^{\mathcal{N}}(E_n) \rightarrow L_n^{\mathcal{N}}(E)$ is a stable motivic equivalence of cofibrant objects.

Denote

$$L_n^{\text{GL}}(E) := L(L_n^{\mathcal{N}}(E)).$$

Then $L(E) = \text{colim}_n L_n^{\text{GL}}(E)$, because L preserves colimits. By Corollary 5.6, L is a left Quillen functor, and hence $L(\varphi_n) : F_n^{\text{GL}}(E_n) \rightarrow L_n^{\text{GL}}(E)$ is a stable equivalence in $\text{Sp}_T^{\text{GL}}(k)$ by [17, 1.1.12]. By Proposition 6.7 $UL(\varphi_n)$ is a stable motivic equivalence in $\text{Sp}_T^{\mathcal{N}}(k)$. Consider a commutative square

$$\begin{array}{ccc} F_n^{\mathcal{N}}(E_n) & \xrightarrow{\beta_n} & U(F_n^{\text{GL}}(E_n)) \\ \varphi_n \downarrow & & \downarrow UL(\varphi_n) \\ L_n^{\mathcal{N}}(E) & \xrightarrow{\gamma_n} & U(L_n^{\text{GL}}(E)) \end{array}$$

with γ_n the counit map. Since $\beta_n, \varphi_n, UL(\varphi_n)$ are stable motivic equivalences, then so is γ_n . It follows from [10, 3.5] that

$$\gamma := \operatorname{colim}_n \gamma_n : E = \operatorname{colim}_n L_n^{\mathcal{N}}(E) \rightarrow UL(E) = \operatorname{colim}_n U(L_n^{\text{GL}}(E))$$

is a stable motivic equivalence, because $\operatorname{Sp}_T^{\mathcal{N}}(k)$ is a weakly finitely generated model category.

Let $\delta : L(E) \rightarrow RL(E)$ be a fibrant resolution of $L(E)$ in $\operatorname{Sp}_T^{\text{GL}}(k)$. Then $U(\delta)$ is a stable motivic equivalence in $\operatorname{Sp}_T^{\mathcal{N}}(k)$ by Proposition 6.7. We see that the composition $E \xrightarrow{\gamma} UL(E) \xrightarrow{U(\delta)} U(RL(E))$ is a stable motivic equivalence for any cofibrant $E \in \operatorname{Sp}_T^{\mathcal{N}}(k)$. Since U plainly reflects stable equivalences between fibrant GL-spectra, (L, U) is a Quillen equivalence by [17, 1.3.16]. This completes the proof of the theorem. ■

We discuss an application of Theorem 6.1 in the next section concerning the localization functor $C_*\mathcal{F}r$ of [15].

7. On the localization functor $C_*\mathcal{F}r$

Throughout this section k is an (infinite) perfect field. As usual, we assume $\operatorname{char} k \neq 2$ whenever we deal with orthogonal or special orthogonal motivic spectra. Recall that $\operatorname{SH}_{\text{nis}}(k)$ is the triangulated category obtained from the local stable homotopy category of sheaves of S^1 -spectra $\operatorname{SH}_{S^1}^{\text{nis}}(k)$ by stabilizing $\operatorname{SH}_{S^1}^{\text{nis}}(k)$ with respect to the endofunctor $\mathbb{G}_m^{\wedge 1} \wedge -$.

Let \mathcal{T} be a triangulated category. Following [1], we define a localization in \mathcal{T} as a triangulated endofunctor $L : \mathcal{T} \rightarrow \mathcal{T}$ together with a natural transformation $\eta : \operatorname{id} \rightarrow L$ such that $L\eta_X = \eta_{LX}$ for any X in \mathcal{T} and η induces an isomorphism $LX \cong LLX$. We refer to L as a **localization functor** in \mathcal{T} . Such a localization functor determines a full subcategory $\operatorname{Ker} L$ whose objects are those X such that $LX = 0$. An object $X \in \mathcal{T}$ is said to be **L -local** if $\eta_X : X \rightarrow LX$ is an isomorphism.

The computation of localization functors and their full subcategories of local objects is enormously hard in practice. In particular, if $\mathcal{T} = \operatorname{SH}_{\text{nis}}(k)$ and \mathcal{S} is the full subcategory of $\operatorname{SH}_{\text{nis}}(k)$ compactly generated by the shifted cones of the arrows

$$pr_X : \Sigma_{S^1}^{\infty} \Sigma_{\mathbb{G}_m^{\wedge 1}}^{\infty} (X \times \mathbb{A}^1)_+ \rightarrow \Sigma_{S^1}^{\infty} \Sigma_{\mathbb{G}_m^{\wedge 1}}^{\infty} X_+, \quad X \in Sm/k,$$

then the Bousfield localization theory in compactly generated triangulated categories says that there exists a localisation functor

$$L_{\mathbb{A}^1} : \mathrm{SH}_{\mathrm{nis}}(k) \rightarrow \mathrm{SH}_{\mathrm{nis}}(k)$$

such that $\mathcal{S} = \mathrm{Ker} L_{\mathbb{A}^1}$. By definition, the Morel–Voevodsky stable motivic homotopy category $\mathrm{SH}(k)$ is the quotient category $\mathrm{SH}_{\mathrm{nis}}(k)/\mathcal{S}$.

A new approach to the classical stable homotopy theory $\mathrm{SH}(k)$ of Morel–Voevodsky [23] was suggested in [15]. This approach has nothing to do with any kind of motivic equivalences and is briefly defined as follows. There exists an explicit localization functor

$$C_*\mathcal{F}r : \mathrm{SH}_{\mathrm{nis}}(k) \rightarrow \mathrm{SH}_{\mathrm{nis}}(k)$$

that first takes a bispectrum E to its naive projective cofibrant resolution E^c and then one sets in each bidegree $C_*\mathcal{F}r(E)_{i,j} := C_*Fr(E_{i,j}^c)$ (we refer the reader to [14] for the definition of $C_*Fr(\mathcal{X})$, $\mathcal{X} \in \mathbb{M}_\bullet$). We should note that the localization functor $C_*\mathcal{F}r$ is isomorphic to the big framed motives localization functor \mathcal{M}_{fr}^b of [14] (see [15] as well). We then define $\mathrm{SH}^{\mathrm{new}}(k)$ as the category of $C_*\mathcal{F}r$ -local objects in $\mathrm{SH}_{\mathrm{nis}}(k)$. By [15, Section 2] $\mathrm{SH}^{\mathrm{new}}(k)$ is canonically equivalent to Morel–Voevodsky’s $\mathrm{SH}(k)$.

The localization functor $C_*\mathcal{F}r$ is also of great utility when dealing with another model for $\mathrm{SH}(k)$, constructed in [15]. This model recovers all motivic bispectra as certain covariant functors on $Fr_0(k)$ taking values in \mathbb{A}^1 -local framed S^1 -spectra. In particular, this model of $\mathrm{SH}(k)$ implies that $\pi_{i,j}^{\mathbb{A}^1}(E)$ -s have more information than just the naive bigraded sheaves. Namely, they are recovered from certain covariant functors $\pi_i^{fr}(E)$ on $Fr_0(k)$ taking values in strictly \mathbb{A}^1 -invariant framed sheaves. Thus the functors $\pi_i^{fr}(E)$ have one index only corresponding to the S^1 -direction (in this way we get rid of the second index). These are reminiscent of the classical stable homotopy groups of ordinary S^1 -spectra. It is therefore useful to think of the $\pi_{i,j}^{\mathbb{A}^1}(E)$ as the richer information “ $\pi_i^{fr}(E)$ ”.

Theorems 6.1 and 6.5 give rise to an equivalent model for the localization functor $C_*\mathcal{F}r$ (see below). It involves smooth algebraic varieties of the form G_{n+k}/G_n , where G_n , $n \geq 0$, is GL_n , SL_{2n} , Sp_{2n} , O_{2n} or SO_{2n} . Below we shall write \mathcal{G} to denote the family $\{G_n\}_{n \geq 0}$. In this paper \mathcal{G} is $\{\mathrm{GL}_n\}_{n \geq 0}$, $\{\mathrm{SL}_{2n}\}_{n \geq 0}$, $\{\mathrm{Sp}_{2n}\}_{n \geq 0}$, $\{\mathrm{O}_{2n}\}_{n \geq 0}$ or $\{\mathrm{SO}_{2n}\}_{n \geq 0}$.

7.1. *Definition:* Let $\mathcal{G} = \{G_k\}_{k \geq 0}$ be a family as above, $n \geq 0$ and $\mathcal{X} \in \mathbb{M}_\bullet$. If $\mathcal{G} = \{GL_k\}_{k \geq 0}$ define

$$Fr^{\mathcal{G},n}(\mathcal{X}) := \operatorname{colim}_{q \geq n} \underline{\operatorname{Hom}}_{\mathbb{M}_\bullet}(\mathbb{P}^{\wedge q}, \mathcal{X} \wedge (G_q/G_{q-n})_+ \wedge T^q).$$

In other words, if we consider the \mathbb{P}^1 -spectrum

$$\mathcal{Y} := (*, \cdot^{-1}, *, \mathcal{X} \wedge (G_n)_+ \wedge T^n, \mathcal{X} \wedge (G_{n+1}/G_1)_+ \wedge T^{n+1}, \mathcal{X} \wedge (G_{n+2}/G_2)_+ \wedge T^{n+1}, \dots)$$

then $Fr^{\mathcal{G},n}(\mathcal{X})$ equals the 0th space of the spectrum $\Theta_{\mathbb{P}^1}^\infty(\mathcal{Y})$. Notice that G_{n+k}/G_n -s incorporated into the definition are all smooth algebraic varieties. In turn, if \mathcal{G} is $\{SL_{2k}\}_{k \geq 0}$, $\{Sp_{2k}\}_{k \geq 0}$, $\{O_{2k}\}_{k \geq 0}$ or $\{SO_{2k}\}_{k \geq 0}$ and $n \geq 0$ is even, then $Fr^{\mathcal{G},n}(\mathcal{X})$ is defined as above if we take the colimit over even q -s.

Using the terminology of [14], we define the (\mathcal{G}, n) -**framed motive** $M_{fr}^{\mathcal{G},n}(\mathcal{X})$ of \mathcal{X} as the Segal S^1 -spectrum associated with the (sectionwise) Γ -space $m \in \Gamma^{\text{op}} \mapsto C_* Fr^{\mathcal{G},n}(\mathcal{X} \wedge m_+)$, where C_* stands for the Suslin complex.

If we want to specify the choice of groups, we write below $C_* Fr^{GL,n}(\mathcal{X})$, $C_* Fr^{SL,2n}(\mathcal{X})$, $C_* Fr^{Sp,2n}(\mathcal{X})$, $C_* Fr^{O,2n}(\mathcal{X})$, and $C_* Fr^{SO,2n}(\mathcal{X})$ (respectively, we write $M_{fr}^{GL,n}(\mathcal{X})$, $M_{fr}^{SL,2n}(\mathcal{X})$, $M_{fr}^{Sp,2n}(\mathcal{X})$, $M_{fr}^{O,2n}(\mathcal{X})$, and $M_{fr}^{SO,2n}(\mathcal{X})$).

Let $\Delta^{\text{op}} Fr_0(k)$ be the category of simplicial objects in $Fr_0(k)$. There is an obvious fully faithful functor $spc : Fr_0(k) \rightarrow Shv_\bullet(Sm/k)$ sending an object $X \in Fr_0(k)$ to the Nisnevich sheaf X_+ . It induces a fully faithful functor

$$spc : \Delta^{\text{op}} Fr_0(k) \rightarrow sShv_\bullet(Sm/k),$$

taking an object $[n] \mapsto Y_n$ to the simplicial Nisnevich sheaf $[n] \mapsto (Y_n)_+$. Denote the image of this functor by \mathcal{T} . Also, we shall write $\vec{\mathcal{T}}$ to denote the motivic spaces which are filtered colimits of objects in \mathcal{T} coming from filtered diagrams in $\Delta^{\text{op}} Fr_0(k)$ under the functor spc .

7.2. **THEOREM:** *Suppose $\mathcal{X} \in \vec{\mathcal{T}}$. Under the notation of Definition 7.1 there is a natural stable local equivalence of S^1 -spectra $\mu : M_{fr}(\mathcal{X}) \rightarrow M_{fr}^{GL,n}(\mathcal{X})$, where $n \geq 0$. If n is even and $\mathcal{G} \in \{SL, Sp, O, SO\}$ then there is also a natural stable local equivalence of S^1 -spectra $\mu : M_{fr}(\mathcal{X}) \rightarrow M_{fr}^{\mathcal{G},n}(\mathcal{X})$.*

Proof. We shall prove the theorem for the case $\mathcal{G} = \{GL_n\}_{n \geq 0}$. The proof for the other choices of \mathcal{G} is similar. Without loss of generality we may assume for simplicity $\mathcal{X} = X_+$, where $X \in Sm/k$. By the proof of Theorem 6.5 there is a

natural stable motivic equivalence of T -spectra

$$\beta : sh^{-n}(\Sigma_T^\infty(\mathcal{X} \wedge T^n)) \rightarrow \mathcal{Y},$$

where $sh^{-n}(\Sigma_T^\infty(\mathcal{X} \wedge T^n)) = (*, {}^{n-1}, *, \mathcal{X} \wedge T^n, \mathcal{X} \wedge T^{n+1}, \dots)$ is the $(-n)$ th shift of $\Sigma_T^\infty(\mathcal{X} \wedge T^n)$ and \mathcal{Y} as in Definition 7.1. Observe that both spectra are Thom spectra with the bounding constant $d \leq 1$ in the sense of [11].

By the proof of [20, 2.13] β is a stable motivic equivalence of \mathbb{P}^1 -spectra, and hence so is

$$\Theta_{\mathbb{P}^1}^\infty(\beta) : \Theta_{\mathbb{P}^1}^\infty(sh^{-n}(\Sigma_T^\infty(\mathcal{X} \wedge T^n))) \rightarrow \Theta_{\mathbb{P}^1}^\infty(\mathcal{Y}).$$

We have,

$$C_*\Theta_{\mathbb{P}^1}^\infty(sh^{-n}(\Sigma_T^\infty(\mathcal{X} \wedge T^n))) = (C_*Fr(\mathcal{X}), C_*Fr(\mathcal{X} \wedge T), C_*Fr(\mathcal{X} \wedge T^2), \dots)$$

and

$$C_*\Theta_{\mathbb{P}^1}^\infty(\mathcal{Y}) = (C_*Fr^{GL,n}(\mathcal{X}), C_*Fr^{GL,n}(\mathcal{X} \wedge T), C_*Fr^{GL,n}(\mathcal{X} \wedge T^2), \dots).$$

Since the map $C_*\Theta_{\mathbb{P}^1}^\infty(sh^{-n}(\Sigma_T^\infty(\mathcal{X} \wedge T^n))) \rightarrow C_*\Theta_{\mathbb{P}^1}^\infty(\mathcal{Y})$ is a stable motivic equivalence, it follows from [11, 5.2] that the map of spaces

$$\nu : C_*Fr(\mathcal{X} \wedge T) \rightarrow C_*Fr^{GL,n}(\mathcal{X} \wedge T)$$

is a local equivalence. By [12, A.1] and the proof of [11, 9.9] both spaces are locally connected. It follows from [14, 6.4] that these are the underlying spaces of (locally) very special Γ -spaces, and so the map of S^1 -spectra

$$\xi : M_{fr}(\mathcal{X} \wedge T) \rightarrow M_{fr}^{GL,n}(\mathcal{X} \wedge T)$$

is a level local equivalence.

Consider a commutative diagram

$$\begin{array}{ccc} M_{fr}(\mathcal{X})_f & \xrightarrow{\beta} & M_{fr}^{GL,n}(\mathcal{X})_f \\ \downarrow & & \downarrow \\ \Omega_{\mathbb{P}^1}(M_{fr}(\mathcal{X} \wedge T))_f & \xrightarrow{\xi_*} & \Omega_{\mathbb{P}^1}(M_{fr}^{GL,n}(\mathcal{X} \wedge T))_f. \end{array}$$

Here f refers to the stable local fibrant replacement of S^1 -spectra and the upper arrow is induced by β . It follows from [14, 7.1] that all spectra are motivically fibrant. Then the map ξ_* is a level weak equivalence of motivically fibrant spectra. The proof of [14, 4.1(2)] shows that the vertical arrows are level weak

equivalences (we also use [11, Section 9]), and hence so is the upper arrow. It follows that the map

$$M_{Fr}(\mathcal{X}) \rightarrow M_{Fr}^{GL,n}(\mathcal{X})$$

is a stable local equivalence, as was to be shown. ■

If $\mathcal{X} \mapsto \mathcal{X}^c$ is the cofibrant replacement functor in the projective motivic model structure in \mathbb{M}_\bullet , then \mathcal{X}^c belongs to $\vec{\mathcal{T}}$ (see [14, Section 10]).

7.3. THEOREM: *Under the assumptions of Theorem 7.2 let $C_*\mathcal{F}r^{\mathcal{G},n}$ be the functor on bispectra taking an $(S^1, \mathbb{G}_m^{\wedge 1})$ -bispectrum E to the bispectrum $C_*\mathcal{F}r^{\mathcal{G},n}(E)$ which is defined in each bidegree as*

$$C_*\mathcal{F}r^{\mathcal{G},n}(E)_{i,j} := C_*Fr^{\mathcal{G},n}(E_{i,j}^c),$$

where E^c is a projective cofibrant resolution of E . Then $C_*\mathcal{F}r^{\mathcal{G},n}$ is an endofunctor on $\text{SH}_{\text{nis}}(k)$ and is naturally isomorphic to the localizing functor

$$C_*\mathcal{F}r : \text{SH}_{\text{nis}}(k) \rightarrow \text{SH}_{\text{nis}}(k)$$

if $\mathcal{G} = \{\text{GL}_k\}_{k \geq 0}$ and n is any non-negative integer, or if $\mathcal{G} \in \{SL, Sp, O, SO\}$ and n is even non-negative. In particular, one has a localizing functor

$$C_*\mathcal{F}r^{\mathcal{G},n} : \text{SH}_{\text{nis}}(k) \rightarrow \text{SH}_{\text{nis}}(k)$$

such that the category of $C_*\mathcal{F}r^{\mathcal{G},n}$ -local objects is $\text{SH}^{\text{new}}(k)$.

Proof. By the Additivity Theorem of [14] $C_*\mathcal{F}r(-, Y)$ and $C_*\mathcal{F}r^{\mathcal{G},n}(-, Y)$ are special Γ -spaces for Y a filtered colimit of simplicial schemes from $\Delta^{\text{op}}Fr_0(k)$. Let F be an S^1 -spectrum such that every entry F_j of F is a filtered colimit of k -smooth simplicial schemes from $\Delta^{\text{op}}Fr_0(k)$. F has a natural filtration $F = \text{colim}_m L_m(F)$, where $L_m(F)$ is the spectrum

$$(F_0, F_1, \dots, F_m, F_m \wedge S^1, F_m \wedge S^2, \dots).$$

Then

$$C_*Fr(F) = C_*Fr(\text{colim}_m L_m(F)) = \text{colim}_m C_*Fr(L_m(F)),$$

where $C_*Fr(L_m(F))$ is the spectrum

$$(C_*Fr(F_0), C_*Fr(F_1), \dots, C_*Fr(F_m), C_*Fr(F_m \otimes S^1), C_*Fr(F_m \otimes S^2), \dots).$$

Similarly, one has

$$C_*\mathcal{F}r^{\mathcal{G},n}(F) = C_*\mathcal{F}r^{\mathcal{G},n}(\text{colim}_m L_m(F)) = \text{colim}_m C_*\mathcal{F}r^{\mathcal{G},n}(L_m(F)),$$

where $C_*Fr^{\mathcal{G},n}(L_m(F))$ is the spectrum

$$(C_*Fr^{\mathcal{G},n}(F_0), C_*Fr^{\mathcal{G},n}(F_1), \\ \dots, C_*Fr^{\mathcal{G},n}(F_m), C_*Fr^{\mathcal{G},n}(F_m \otimes S^1), C_*Fr^{\mathcal{G},n}(F_m \otimes S^2), \dots).$$

Observe that

$$sh^n C_*Fr(L_m(F)) = M_{fr}(F_m) \quad \text{and} \quad sh^n C_*Fr^{\mathcal{G},n}(L_m(F)) = M_{fr}^{\mathcal{G},n}(F_m).$$

By Theorem 7.2 the natural map $M_{fr}(F_m) \rightarrow M_{fr}^{\mathcal{G},n}(F_m)$ is a stable local equivalence, and hence so is $C_*Fr(L_m(F)) \rightarrow C_*Fr^{\mathcal{G},n}(L_m(F))$. Thus the natural map $C_*Fr(F) \rightarrow C_*Fr^{\mathcal{G},n}(F)$ is a stable local equivalence of spectra. Thus if E is a bispectrum then the natural map of bispectra

$$C_*Fr(E) \rightarrow C_*Fr^{\mathcal{G},n}(E)$$

is a level stable local equivalence. The fact that $C_*Fr^{\mathcal{G},n}$ is an endofunctor on $\mathrm{SH}_{\mathrm{nis}}(k)$ is obvious as well as that both functors are isomorphic on $\mathrm{SH}_{\mathrm{nis}}(k)$. This completes the proof. \blacksquare

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