Global well-posedness of 2D stochastic Burgers equations with multiplicative noise

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Abstract

In this article, we study 2D stochastic Burgers equations driven by linear multiplicative noise, and with non-periodic boundary conditions. We first apply Galerkin approximation method to show the local existence and uniqueness of strong solutions, we then establish the global well-posedness for strong solutions by utilizing the maximum principle.

MSC: Primary 60H15; Secondary 76S05.
Keywords: Stochastic 2D Burgers equations; global well-posedness; Galerkin approximation; maximum principle

1 Introduction

This article is concerned with 2D Stochastic Burgers equations (SBEs) prescribed on a smooth, bounded, open domain $D \subset \mathbb{R}^2$. For arbitrarily fixed $T > 0$, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a given stochastic basis. Set $W := \sum_{k=1}^{\infty} \sigma_k B_k(t)$, for $t \in [0,T]$, $\sigma_k \in \mathbb{R}$ with $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, where $\{B_k(t)\}_{t\in[0,T]}, k \in \mathbb{N}$ is a sequence of independent, one-dimensional $\{\mathcal{F}_t\}_{t\in[0,T]}$-Wiener processes. On the other, let us denote $\Delta := \partial_{x_1}^2 + \partial_{x_2}^2$ the Laplace operator, and $\nabla := (\partial_{x_1}, \partial_{x_2})$ the gradient operator. We consider the Cauchy problem for the following 2D SBE driven by linear multiplicative noise, and subject to the Dirichlet boundary condition

$$du(t) = \nu\Delta u dt - (u \cdot \nabla)u dt + u(t) \circ dW(t), \tag{1.1}$$

$$u(t,x) = 0, \qquad t > 0,\ x = (x_1, x_2) \in \partial D, \tag{BC}$$

$$u(0,x) = u_0(x), \qquad x = (x_1, x_2) \in D, \tag{IC}$$

for the unknowns 2D vector-valued random fields $u(t,x) = (u_1(t,x), u_2(t,x)) \in \mathbb{R}^2$ for $(t,x) \in [0,T] \times D$, where the parameter $\nu > 0$ stands for the viscosity and $\circ$ denotes the Stratonovich integral. The mathematical study of the Burgers equation was originated in a series of articles (chronologically) by Forsyth [12], Beteman [3], and Burgers [8]. The case of scalar SBEs (i.e., $\mathbb{R}$-valued random fields u) has been pretty well studied by Bertini, Cancrini and Jona-Lasinio in [4] and Da Prato, Debussche and Temam in [9], just mention a few.

Towards the case of higher dimensional inviscid SBEs, stationary solutions and stationary distributions were constructed by Iturriaga and Khanin in [15]. Based on the stochastic version of Lax formula, Gomes,

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†Research partially supported by NNSF of China(Grant No. 11971077), Chongqing Key Laboratory of Analytic Mathematics and Applications, Chongqing University, Natural Science Foundation Project of CQ(Grant No. cstc2020jcyj-msxmX0441), Fundamental Research Funds for the Central Universities(Grant No. 2020CDJ-LHZZ-027).
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Iturriaga, Khanin and Padilla in [13] proved the convergence of stationary distributions for the randomly forced multi-dimensional Burgers equations when viscosity tends to zero. Utilizing some inventive techniques, Brzezniak, Goldys and Neklyudov [5] established the global well-posedness of multidimensional Burgers equations with additive noise effecting only on one coordinate. Furthermore, the asymptotic behavior of solutions is studied when the viscosity tends to zero. For the potential case, one can see [7, 16, 11] and other references therein.

In the present work, we consider the global well-posedness of 2D SBEs with Dirichlet boundary conditions driven by linear multiplicative noise. Here, we should point out that the absence of the incompressible property and high nonlinearity of 2D SBEs bring difficulties to establish *a priori* estimates even in L^2 space. To overcome the difficulties, we heavily rely on the random version of maximum principle (Lemma 4.1) and an argument of compactness and regularity of the solutions to 2D SBEs (see the proof of Theorem 4.1). In the forthcoming work by the same authors, the global well-posedness of 2D SBEs with periodic boundary conditions is established without the help of Poincaré's inequality, hence requires more delicate techniques.

In a recent paper [19], Zhang, Zhou, Guo and Wu studied 3D stochastic Burgers equations with the noise perturbing only one coordinate, with the initial data lying in $L^\infty(D)$. The present article aims to extend the noise to all the coordinates and drops the assumption that the initial data should lie in $L^\infty(D)$. Due to the dimension is 2 and the noise is multiplicative, instead of utilizing the contraction principle argument, we first do a martingale type transform, then adopt the techniques from partial differential equations (PDEs), that is, we first derive energy estimates in a more regular spaces, then establish the local existence of the solutions by applying comparison principle. Also, since the noise perturbs all coordinates, we need to adopt a new version of maximum principle (see Lemma 4.1), and derive *a priori* estimates to help us obtain the global well-posedness.

The maximum principle stated in Lemma 4.1 is the key tool to establish the global existence of the strong solution to (1.1) with (BC)-(IC). It is well known that the maximum principle should be applied to the classical solutions to differential equations. However, for stochastic partial differential equations and stochastic ordinary differential equations, there is no classical solution. Therefore, we can not consider the global well-posedness for 2D SBEs with nonlinear multiplicative noise. The novelty of the present paper is that we apply the maximum principle to the random Galerkin approximations. Then, utilizing the classic compactness arguments shown in Lemma 2.1 and Lemma 2.2, we obtain a subsequence of the solutions converging almost everywhere with respect to time to the solutions to (1.1) with (BC)-(IC). To refine the almost everywhere result, we make use of the continuity of the local solutions with respect to time, and achieve the result that *a prior* estimates hold for any time.

Finally, we would like to point out the differences between [5] and the present work. Firstly, the boundary conditions in [5] are periodic, while in our present article the boundary conditions are Dirichlet. Secondly, the noise considered in [5] is additive and the noise exists only in one coordinate, while, in our article we deal with (linear) multiplicative noise which perturbs on all the coordinates. Finally, in [5], the proof of local existence of solutions relies on the semigroup method, while here we use the Galerkin approximation approach.

The rest of this article is organized as follows. Some preliminaries are presented in Section 2, local well-poseness result of the stochastic system is stated and proved in Section 3. In Section 4, we establish the global well-posedness of the 2D stochastic Burgers equation.

2 Preliminaries

We first introduce the notations that will be used throughout this article. For $p \in \mathbb{N}^+$, let $L^p(D;\mathbb{R}^2)$ be the vector-valued L^p-space in which the norm is denoted by $|\cdot|_p$. When $p=2$, denote by $\mathbb{H} := L^2(D;\mathbb{R}^2)$ and its associated norm and inner product are $|\cdot|_2$ and $\langle\cdot,\cdot\rangle$, respectively. Moreover, when $p=\infty$, $L^\infty(D;\mathbb{R}^2)$ stands for the collection of vector-valued functions which are essentially bounded on D and the corresponding norm is denoted by $|\cdot|_\infty$.

For $m \in \mathbb{N}^+$, $(W^{m,p}(D), \|\cdot\|_{m,p})$ is the classical Sobolev space. When $p=2$, denote by $\mathbb{H}^m(D) = W^{m,2}(D)$, and

$$\|u\|_m^2 = \sum_{0\le|\delta|\le m} \int_D |D^\delta u|^2 dx,$$

where $\delta := (\delta_1, \delta_2)$ is a multi-index with nonnegative integers δ_1, δ_2, and $|\delta| = \delta_1 + \delta_2$. It is well known that $(H^m(D), \|\cdot\|_m)$ is a Hilbert space. Let $C_c^\infty(D)$ be the space of all infinitely differentiable functions on D with compact support. Let $W_0^{1,p}$ be the closure of $C_c^\infty(D)$ in $W^{1,p}(D)$. Set $\mathbb{H}_0^1 = W_0^{1,2}$ and let $\mathbb{H}^{-1}$ be the dual space of $\mathbb{H}_0^1$.

In this article, we simply deal with the case that the viscosity $\nu = 1$. In fact, ν can be any strictly positive real. Denote by $A := -\Delta$, then $A : \mathbb{H}_0^1 \to \mathbb{H}^{-1}$ and $D(A) = [\mathbb{H}^2 \cap \mathbb{H}_0^1]^2$, A is a positive self-adjoint operator with discrete spectrum in $\mathbb{H}$, that is, there exists an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ in $\mathbb{H}$ and a sequence of increasing real values $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $Ae_k = \lambda_k e_k$.

For any $u \in L^2(D)$, denote by $u_k = \langle u, e_k\rangle$. Given $s \in \mathbb{R}$, the fractional power $(A^s, D(A^s))$ of the operator $(A, D(A))$ is defined by

$$A^s u = \sum_{n=1}^{\infty} \lambda_n^s u_n e_n, \text{ where } u = \sum_{n=1}^{\infty} u_n e_n;\ D(A^s) = \{u = \sum_{n=1}^{\infty} u_n e_n \big| \sum_{n=1}^{\infty} \lambda_n^{2s} |u_n|^2 < \infty\}.$$

We then set $\mathbb{H}^s := D(A^{s/2})$ and denote by $\|\cdot\|_s$ the seminorm $|A^{s/2}\cdot|_2$.

Next, let us introduce strong solutions to (1.1) with (BC)-(IC).

Definition 2.1 (Local strong solutions)**.** Let the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$ be given as before. Suppose u_0 is a $\mathbb{H}^1$-valued, $\mathcal{F}_0$-measurable random variable.

(i) A pair (u, τ) is a **local strong pathwise solution** to (1.1) with (BC)-(IC) if τ is a strictly positive stopping time and $u(\cdot \wedge \tau)$ is an $\mathcal{F}_t$-adapted process in $\mathbb{H}^1$ so that

$$u(\cdot\wedge\tau) \in L^2\left(\Omega; C([0,\infty); \mathbb{H}^1)\right), \qquad u\mathbb{1}_{t\leq\tau} \in L^2\left(\Omega; L^2_{\text{loc}}([0,\infty); \mathbb{H}^2)\right); \tag{2.1}$$

and for any $t \geq 0$, the following identity holds in $\mathbb{H}$,

$$u(t\wedge\tau) - \int_0^{t\wedge\tau} \Delta u(s)ds + \int_0^{t\wedge\tau} (u\cdot\nabla u)(s)ds = u_0 + \sum_{k=1}^{\infty}\int_0^{t\wedge\tau} \sigma_k u(s)\circ dB_k(s).$$

(ii) Strong pathwise solutions of (1.1) are said to be **unique** up to a random positive time $\tau > 0$ if given any pair of solutions $(u^1, \tau), (u^2, \tau)$, which coincide at $t = 0$ on the event $\tilde{\Omega} = \{u^1(0) = u^2(0)\} \subset \Omega$, then

$$\mathbb{P}(\mathbb{1}_{\tilde{\Omega}}\big(u^1(t\wedge\tau) - u^2(t\wedge\tau)\big) = 0; \forall t \geq 0) = 1.$$

Definition 2.2 (Maximal and global strong solutions)**.**

(i) Let ξ be a positive random variable. We say that (u, ξ) is a **maximal pathwise strong solution** if (u, τ) is a local strong pathwise solution for each $\tau < \xi$ and $\sup_{t\in[0,\xi)} \|u\|_1 = \infty$ almost surely on the set $\{\xi < \infty\}$.

(ii) If (u, ξ) is a maximal pathwise strong solution and $\xi = \infty$ a.s., then we we say the solution is **global**.

Let $\alpha : [0,\infty] \times \Omega \to \mathbb{R}$ be the solution to the following Stratonovich stochastic differential equation:

$$d\alpha(t) = \sum_{k=1}^{\infty} \sigma_k \alpha(t) \circ dB_k(t), \qquad \text{for } t \geq 0, \text{ and } \alpha(0) = 1. \tag{2.2}$$

Applying Itô's formula, we have for any $t \geq 0$, $\alpha(t) = \exp\{\sum_{k=1}^{\infty} \sigma_k B_k(t)\}$. By Novikov's condition and Doob's maximal inequality, we know that for any $T > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\alpha(t)|\right] < \infty.$$

We now make change of variables by $v(t) = \alpha(t)u(t)$, then $v(t)$ satisfies the following equation:

$$dv(t) = \Delta v(t)dt + \alpha(t)^{-1}(v(t)\cdot\nabla)v(t)dt, \tag{2.3a}$$

$$v(t,x) = 0, \qquad (t,x) \in [0,T] \times \partial D; \tag{2.3b}$$

$$v(0,x) = u_0(x), \qquad x = (x_1, x_2) \in D. \tag{2.3c}$$

We first consider the Galerkin approximation of (2.3). For $n \in \mathbb{N}$, let P_n be the projection from $\mathbb{H}$ onto the subspace expanded by $\{e_1, e_2, \dots, e_n\}$, that is, for $u \in \mathbb{H}$,

$$P_n(u) = P_n \left(\sum_{k=1}^{\infty} \langle u, e_k \rangle e_k \right) = \sum_{k=1}^{n} \langle u, e_k \rangle e_k.$$

Then we obtain the Galerkin approximation of (2.3) as follows:

$$dv_n(t,x) = \Delta v_n(t,x)dt - \alpha(t)^{-1} P_n[(v_n \cdot \nabla) v_n(t,x)]dt, \qquad (t,x) \in [0,T] \times D, \tag{2.4a}$$

$$v_n(t,x) = 0, \qquad (t,x) \in [0,T] \times \partial D; \tag{2.4b}$$

$$v_n(0,x) = u_n(0,x), \qquad x = (x_1, x_2) \in D. \tag{2.4c}$$

Since (2.4) is a locally Lipschitz system of stochastic ODEs, there exists the unique local solution v_n to (2.4) with $\tau_{n,\omega}$ as the maximal existence time of v_n. Obviously, $v_n \in C([0, \tau_{n,\omega}] \times D)$. To end this section, we present Aubin-Lions lemma and Lions-Magenes lemma as follows. One can refer to [18] for proof details.

Lemma 2.1 (Aubin-Lions, [1]). *Let B_0, B, B_1 be Banach spaces such that B_0, B_1 are reflexive and $B_0 \overset{c}{\subset} B \subset B_1$. Define for $0 < T < \infty$,*

$$X := \left\{ h \middle| h \in L^2([0,T]; B_0), \frac{dh}{dt} \in L^2([0,T]; B_1) \right\}.$$

Then X is a Banach space equipped with the norm $|h|_{L^2([0,T];B_0)} + |h'|_{L^2([0,T];B_1)}$. Moreover, $X \overset{c}{\subset} L^2([0,T]; B)$.

Lemma 2.2 (Lions-Magenes, [17]). *Let V, H, V' be three Hilbert spaces such that $V \subset H = H' \subset V'$, where H' and V' are the dual spaces of H and V respectively. Suppose $u \in L^2(0,T;V)$ and $u' \in L^2(0,T;V')$. Then u is almost everywhere equal to a function continuous from $[0,T]$ into H.*

3 Local well-posedness of stochastic Burgers equations

In this section, we use the Galerkin approximation method to show the local existence of the soltuions to (2.3).

Proposition 3.1. *Assume that the initial data $u_0 \in \mathbb{H}^1$ is $\mathcal{F}_0$-measurable, there exists a random variable $\tau > 0$ such that the unique strong solution v to the equation (2.3) on the interval $[0, \tau]$ satisfies*

$$\sup_{t \in [0,\tau]} \|v(t)\|_1^2 + \int_0^\tau \|v(t)\|_2^2 dt < \infty, \ \mathbb{P}\text{-a.s. } \omega \in \Omega. \tag{3.1}$$

Moreover, v is Lipschitz continuous with respect to the initial data in $\mathbb{H}^1$.

Proof. For any $t \in (0, \tau_{n,\omega})$, taking inner product of (2.4) with $-\Delta v_n$ in $L^2([0,t] \times D)$, then by Hölder's inequality, Young's inequality, interpolation inequality and Sobolev imbedding theorem,

$$\begin{aligned}
\|v_n(t)\|_1^2 + 2\int_0^t \|v_n(s)\|_2^2 ds \le & \|u_0\|_1^2 + \int_0^t \alpha^{-1}(s) \int_D |(v_n \cdot \nabla) v_n(s,x)| |\Delta v_n(s,x)| dx ds \\
\le & \|u_0\|_1^2 + \varepsilon \int_0^t \|v_n(s)\|_2^2 ds + C_\varepsilon \int_0^t \alpha^{-2}(s) |\nabla v_n(s)|_3^2 |v_n(s)|_6^2 ds \\
\le & \|u_0\|_1^2 + \varepsilon \int_0^t \|v_n(s)\|_2^2 ds + C \int_0^t \alpha^{-4}(s) |\|v_n(s)\|_1^2 |v_n(s)|_6^4 ds \\
\le & \|u_0\|_1^2 + \varepsilon \int_0^t \|v_n(s)\|_2^2 ds + C \int_0^t \alpha^{-4}(s) |\|v_n(s)\|_1^6 ds.
\end{aligned}$$

For $t \in [0,1]$, we have

$$\|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds \leq \|u_0\|_1^2 + C\int_0^t \left[\sup_{t\in[0,1]} \alpha^{-4/3}(s)\|v_n(s)\|_1^2\right]^3 ds$$
$$=: \|u_0\|_1^2 + \int_0^t \left[K\|v_n(s)\|_1^2\right]^3 ds. \tag{3.2}$$

Applying comparison theorem (see Theorem III-5-1 in [14]), with $t_* := \frac{1}{2K^3\|u_0\|_1^4}$, one can get

$$\|v_n(t)\|_1^2 \leq \frac{\|u_0\|_1^2}{[1-2tK^3\|u_0\|_1^4]^{1/2}} = \frac{\|u_0\|_1^2}{[1-tt_*^{-1}]^{1/2}}. \tag{3.3}$$

Hence, for any n, the estimate in (3.3) rules out the blowup of v_n in $\mathbb{H}^1$ before the time t_*. We can choose $\tau(\omega) := \frac{t_*(\omega)}{2} > 0$ such that $\tau(\omega)$ does not depend on $n \in \mathbb{N}$. It follows from (3.2) and (3.3) that v_n are uniformly bounded is $L^\infty([0,\tau];\mathbb{H}^1) \cap L^2([0,\tau];\mathbb{H}^2)$. Now back to (2.4), by Hölder inequality and Sobolev imbedding theorem,

$$|\partial_s v_n|_2 \leq \alpha^{-1}|v_n\cdot\nabla v_n|_2 + |\Delta v_n|_2 \leq \alpha^{-1}|v_n|_\infty|\nabla v_n|_2 + \|v_n\|_2 \leq c\alpha^{-1}\|v_n\|_1^{3/2}\|v_n\|_2^{1/2} + \|v_n\|_2, \tag{3.4}$$

where c is a constant that is independent of n, s. Thus, $\partial_s v_n$ is uniformly bounded in $L^2([0,\tau];\mathbb{H})$. By Lemma 2.1 and Lemma 2.2, there exists a subsequence of v_n, which, for convenience, is still denoted by v_n, such that v_n converges to v in $L^2([0,\tau];\mathbb{H}^1)$ and $v \in C([0,\tau];\mathbb{H}^1)$. Following a standard argument, it can be verified that v is the local strong solution to (2.3) and the estimate (3.1) follows analogously from (3.2) and (3.3).

It remains to show the uniqueness. Let v_1, v_2 be two strong solutions to (2.3) with $v_1(0) = v_2(0) = u_0$, and let $\bar{v} = v_1 - v_2$, then we have

$$\begin{aligned}\frac{1}{2}\partial_s\|\bar{v}\|_1^2 + \|\bar{v}\|_2^2 \leq& \alpha^{-1}\langle \bar{v}\cdot\nabla v_1, \Delta\bar{v}\rangle + \alpha^{-1}\langle v_2\cdot\nabla\bar{v}, \Delta\bar{v}\rangle\\ \leq& \alpha^{-1}\|\bar{v}\|_2\|\bar{v}\|_1^{1/2}\|\bar{v}\|_2^{1/2}\|v_1\|_1 + \alpha^{-1}\|v_2\|_1\|\bar{v}\|_1\|\bar{v}\|_2\\ \leq& \varepsilon\|\bar{v}\|_2^2 + c\alpha^{-4}\|\bar{v}\|_1^2\|v_1\|_1^4 + c\alpha^{-2}\|v_2\|_1^2\|\bar{v}\|_1^2.\end{aligned}$$

By Gronwall's inequality, with $\bar{v}(0) = 0$, $\|\bar{v}(s)\|_1 = 0$ for $s \in [0,\tau]$. The Lipschitz continuity of the local strong solution with respect to the initial data in $\mathbb{H}^1$ also follows from the above estimate. □

4 Global well-posedness of stochastic Burgers equations

To establish the global well-posedness, we utilize the maximal principle stated as follows:

Lemma 4.1. *If v_n is a solution to (2.4) on the time interval $[0,t]$, then* $\sup_{s\in[0,t]} |v_n(s)|_\infty \leq |v_n(0)|_\infty$.

Proof. For any $\beta > 0$, set $f(s,x) := e^{-\beta s}v_n(s,x)$ for any $s \in [0,t]$ and $x \in D$. Taking inner product of (2.4) with v_n on both sides gives that

$$\partial_s|v_n(s)|^2 + 2\alpha(s)^{-1}v_n(s)\cdot\nabla|v_n(s)|^2 - 2(\Delta v_n\cdot v_n)(s) = 0.$$

With $|v_n(s)|^2 = |f(s)|^2e^{2\beta s}$ and $2\Delta f(s)\cdot f(s) = \Delta|f(s)|^2 - 2|\nabla f|^2$, we get that

$$\partial_s|f(s)|^2 + 2\beta|f(s)|^2 + 2e^{\beta s}\alpha(s)^{-1}f(s)\cdot\nabla|f(s)|^2 - \Delta|f(s)|^2 + 2|\nabla f(s)|^2 = 0. \tag{4.1}$$

Note that if $|f(s,x)|$ achieves the local maximum for $(s,x) \in (0,t]\times D$, then the left-hand side of (4.1) is strictly positive unless $|f(t,x)| \equiv 0$. Therefore, $|f(s)|_\infty \leq |f(0)|_\infty$, and this yields that

$$|v_n(s)|_\infty \leq e^{\beta s}|v_n(0)|_\infty, \qquad \text{for any } s \in (0,t].$$

The result follows by letting β go to 0. □

Theorem 4.1. *For any $\mathcal{F}_0$-adapted initial data $u_0 \in \mathbb{H}^1$ and any $T > 0$, there exists a unique global strong solution v to (2.3) in the sense of Definition 2.2. Furthermore, $v \in C([0,T];\mathbb{H}^1) \cap L^2([0,T];\mathbb{H}^2)$ and v is Lipschitz continuous with respect to initial data in $\mathbb{H}^1$.*

Proof. Taking inner product of (2.4) with $-\Delta v_n$ in $\mathbb{H}$ and applying Young's inequality gives that

$$\partial_t \|v_n\|_1^2 \leq 2\alpha(t)^{-1} \int_D (v_n \cdot \nabla) v_n \Delta v_n dx - 2\|v_n\|_{\mathbb{H}^2}^2 \leq c\alpha^{-2}(t)|v_n|_\infty^2 \|v_n\|_1^2 - \|v_n\|_2^2. \tag{4.2}$$

By Gronwall's inequality and Lemma 4.1, we have

$$\|v_n(t)\|_1 + \int_0^t \|v_n(s)\|_2^2 ds \leq \|u_0\|_1^2 \exp\left\{ c\|u_0\|_1^2 \int_0^t \alpha(s)^{-2} ds \right\}. \tag{4.3}$$

Now let τ be the maximum existence time of the local strong solution v to (2.3). Hence, for any $\hat{\tau} < \tau$, $\{v_n\}$ is uniformly bounded in $L^2([0,\hat{\tau}];\mathbb{H}^2)$, by Lemma 2.1 and Lemma 2.2, there exists a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, converging to v in $L^2([0,\hat{\tau}];\mathbb{H}^1)$. Hence, using a bootstrapping argument, $v_n(t) \to v(t)$ in $\mathbb{H}^1$ almost every $t \in [0,\tau)$. Furthermore, with (4.3), we obtain that for almost every $t \in [0,\tau)$,

$$\|v(t)\|_1^2 \leq \|u_0\|_1^2 \exp\left\{ c\|u_0\|_1^2 \int_0^t \alpha(s)^{-2} ds \right\}. \tag{4.4}$$

Note that $v(t)$ is continuous with respect to time t in $\mathbb{H}^1$. Hence, (4.4) holds for any $t \in [0,\tau)$. If $\tau < \infty$, then $\limsup_{t\to\tau-} \|v\|_1 = +\infty$, which is contradicting with (4.4). Therefore, $\tau = \infty, \mathbb{P}$-a.s.. Hence, we obtain the global existence for the strong solution v of (2.3). The uniqueness result is proved in Proposition 3.1. □

Acknowledgements We are very grateful to Prof. Zdzislaw Brzezniak for his stimulation and encouragement. We also express our heartfelt thanks to the anonymous referees who devoted much time to reading the article and provided many insightful suggestions.

References

[1] J. Aubin, *Un théorème de compacité*, C. R. Acad. Sci. Paris, 256 (1963), 5042–5044.

[2] Y. Bakhtin, L. Li: *Thermodynamic Limit for Directed Polymers and Stationary Solutions of the Burgers Equation,* Comm. Pure Appl. Math., 72 (2019), 0536–0619.

[3] H. Beteman: *Some recent researches of the motion of fluid*, Monthly Weather Rev., 43 (1915), 163–170.

[4] L. Bertini, N. Cancrini, G. Jona-Lasinio: *The stochastic Burgers equation*, Commun. Math. Phys., 165 (1994), 211–232.

[5] Z. Brzezniak, B. Goldys, M. Neklyudov: *Multidimensional stochastic Burgers equation,* SIAM J. Math. Anal., 46 (2014), 871–889.

[6] A. Budhiraja, P. Dupuis: *A variational representation for positive functionals of infinite dimensional Brownian motion,* Probab. Math. Statist., 20 (2000), 39–61.

[7] A. Boritchev: *Multidimensional potential Burgers turbulence*, Commun. Math. Phys., 342 (2016), 441–489.

[8] J.M. Burgers: *Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion*, Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1., 17 (1939), 1–53.

[9] G. Da Prato, A. Debussche, R. Temam: *Stochastic Burgers' equation,* NoDEA, 1 (1994), 389–402.

[10] A. Dembo, O. Zeitouni, *Large Deviation Techniques and Applications*, Jones and Bartlett, Boston, 1993.

[11] W. E, K. Khanin, A. Mazel, Ya. Sinai: *Invariant measures for Burgers equation with stochastic forcing,* Ann. Math., 151 (2000), 877–900.

[12] A.R. Forsyth: *Theory of differential equations,* Vol. 6, Cambridge University Press, Cambridge, 1906.

[13] D. Gomes, R. Iturriaga, K. Khanin, P. Padilla: *Viscosity limit of stationary distributions for the random forced Burgers equation,* Moscow Mathematical Journal, 5 (2005), 613–631.

[14] P. F. Hsieh, Y. Sibuya, *Basic theory of ordinary differential equations*, Higher Education Press, Beijing, 2007.

[15] R. Iturriaga and K. Khanin: *Burgers turbulence and random Lagrangian systems,* Commun. Math. Phys., 232 (2003), 377–428.

[16] K. Khanin, K. Zhang: *Hyperbolicity of minimizers and regularity of viscosity solutions for a random Hamilton-Jacobi equation,* Commun. Math. Phys., 355 (2017), 803–837.

[17] J. Lions, B. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Springer-Verlag, New York, 1972.

[18] R. Temam, *Navier-Stokes equations, Theory and Numerical Analysis*, reprint of 3rd edition, AMS 2001.

[19] R. Zhang, G. Zhou, B. Guo, and J.-L. Wu, *Global well-posedness and large deviations for 3D stochastic Burgers equations*, Z. Angew. Math. Phys. (2020) 71:30.